Minimax designs for approximately linear regression

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Received 12 September 1990; revised manuscript received 25 March 1991
Recommended by A.S. Hedayat

Abstract: We consider the approximately linear regression model $E[y | x] = z^T(x) \theta + f(x)$, $x \in S$, where $f(x)$ is a non-linear disturbance restricted only by a bound on its $L_p(S)$ norm, and where $S$ is the design space. For loss functions which are monotonic functions of the mean squared error matrix, we derive a theory to guide in the construction of designs which minimize the maximum (over $f$) loss. We then specialize to the case $z^T(x) = (1, x^T)$, so that the fitted surface is a plane. In this case we give minimax designs for loss functions corresponding to the classical D-, A-, E-, Q- and G-optimality criteria.

AMS Subject Classification: Primary 62K05; secondary 62F35, 62G05.

Key words and phrases: Robustness; regression designs; minimax mean squared error; linear regression.

1. Introduction and summary

We consider the construction of designs for multiple linear regression, robust against unspecified contamination of the response function. The experimenter fits, by least squares, the model

$$E[y | x] = z^T(x) \theta \quad (1.1)$$

Here, the regressors $z \in \mathbb{R}^p$ are given functions of $x$, with $x$ varying freely over a design space $S \subset \mathbb{R}^p$. The problem is to choose $n$, not necessarily distinct, design points $x_i \in S$. The precision of the estimate $\hat{\theta}$, determined from observations $(y_i, x_i)$, is to be robust, in a minimax sense, against departures from (1.1) in the true model

$$E[y | x] = z^T(x) \theta + f(x), \quad f \in \mathcal{F} \quad (1.2)$$

The class $\mathcal{F}$ is an $L_2$-neighbourhood, specified at (1.5) below. We assume additive, uncorrelated errors with common variance $\sigma^2$.

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* Research supported by the Natural Sciences and Engineering Research Council of Canada.
The fitting of (1.1), when in fact (1.2) gives the true response, admits a bias com-
ponding the natural variation of $\hat{\theta}$. We thus consider loss functions which are
monotonic functions of the mean squared error matrix of $\hat{\theta}$. We derive a theory to
guide in the construction of designs which minimize the maximum loss, as $f$ ranges
over $\mathcal{F}$. We then present explicit designs in the case that the fitted response is a
plane:
\[ z^T(x) = (1, x^T), \quad z^T(x)\theta = \theta_0 + \sum_{j=1}^q \theta_j x_j, \]
and the loss corresponds to one of the D-, A-, E-, Q-, or G-optimality criteria. The
first three of these require us to minimize (the maximum of) the determinant, trace,
or largest characteristic root, respectively, of the MSE matrix. For the fourth we
minimize the maximum of the integrated (over $S$) MSE of the estimated responses
$\tilde{y}(x) = z^T(x)\hat{\theta}$, while for the fifth we minimize the maximum (over $S$ and $\mathcal{F}$) MSE
of $\tilde{y}(x)$.

Box and Draper (1959) made apparent the dangers of designing a regression ex-
periment which assumes that (1.1) is exactly correct. By analyzing the relative im-
portance of errors due to bias, and to variance, they found that very small
deviations from (1.1) can eliminate any supposed gains arising from the use of a
design which minimizes variance alone.

Beginning with Box and Draper (1959), designs for versions of (1.2) have
been constructed in a series of papers. These differ in the class $\mathcal{F}$, the design
space, the regressors, and in the loss functions used. Marcus and Sacks (1976),
take
\[ \mathcal{F} = \{ f \mid |f(x)| \leq \phi(x) \forall x \in S \}, \]
with various assumptions being made about $\phi$. The optimal designs constructed in
these papers appear to be quite sensitive to the assumed form of $\phi$.

Huber (1975) takes
\[ \mathcal{F} = \left\{ f \mid \| f \|_2 := \left( \int_S f^2(x) \, dx \right)^{1/2} \leq \eta, \quad \int_S z(x)f(x) \, dx = 0 \right\}. \]
The radius $\eta$ is assumed known. The second condition in (1.5) ensures the iden-
tifiability of $\theta$. Huber assumes a simple linear regression model, with loss corre-
sponding to Q-optimality as defined above. Only absolutely continuous, hence
approximate, design measures are considered. It is a consequence of the present
paper that such designs are optimal in the class of all design measures.

The contaminating class (1.5) has been criticized (Marcus and Sacks (1976), Li
and Notz (1982)), evidently as being too full. It is claimed in these papers, and
proved in Section 2 below, that any implementable and non-randomized, hence
discrete, design has infinite maximum loss in the $\mathcal{F}$ of (1.5). The $\mathcal{F}$ of (1.4) seems
to be rather thin, however, since it seems invariably to lead to 'robust' designs, all
of whose mass is concentrated at a small number of, generally extreme, points in
the design space. Note that such designs:
(1) allow little or no opportunity to test the validity of the fitted model;
(2) also have infinite maximum risk in (1.5).
It is to be understood that the continuous designs constructed in this paper
will be approximated by discrete designs in practice. Our attitude is that an approxima-
tion to a design which is robust against more realistic alternatives is preferable to
an exact solution in a neighbourhood which is unrealistically sparse.
For further discussion of continuous regression designs and approximating
designs, see Fedorov (1972), Kiefer (1973), Atkinson (1982).
An alternate, and perhaps more satisfactory, approach is to employ a randomized
design. That is, the design points actually used by the experimenter are chosen ran-
domly from an optimal design density. In game theory terminology, Nature makes
a choice of \( f \) after seeing the Statistician’s choice of a design density, but before see-
ing the actual design points. The expected maximum loss is then finite, and is
minimized by the designs constructed in this paper.
For other approaches to these and related problems, see Stigler (1971), Atwood
(1971), Kiefer (1973), Andrews and Herzberg (1979), Sacks and Ylvisaker (1984),
Notz (1989), Wiens (1991), and the review articles Herzberg (1982), Atkinson
The organization of this paper is as follows. In Section 2 we set out the notation,
and establish some general properties of admissible designs for the model defined
by (1.2) and (1.5). We identify a least favourable, parametric subset of \( \mathcal{F} \). In Section
3, this theory is applied to (1.3), and minimax invariant designs are constructed for
the five loss functions defined above. For D-, Q- and G-optimality we find saddle-
point solutions for all \( n \) and \( \eta \); for A- and F-optimality we show that saddlepoint
solutions exist only for small \( \left( O(n^{-1/2}) \right) \) values of \( \eta \). We outline a method for
determining the minimax solutions in the remaining cases.
It is interesting to note that the D- and G-optimal designs constructed in Section
3 are distinct. This is then in contrast to the case \( \eta = 0 \), in which the D- and G-
optimal designs, for minimizing variance alone, coincide by virtue of the celebrated

2. General theory

Consider the model given by (1.2) and (1.5), where it is assumed that \( S \) has been
linearly transformed in order that
\[
\int_S dx = 1.
\]
The design points \( x_i \) are the atoms of a design measure \( \xi \) on \( S \). If the design matrix
Z has i-th row \( z^T(x_i) \), then we may write \( B = n^{-1}Z^T Z \) and \( Z^T y \) in terms of \((\xi, f)\).

Put

\[
B = B(\xi) = \int_S z(x)z^T(x)\,d\xi(x), \quad b = b(f, \xi) = \int_S z(x)f(x)\,d\xi(x).
\]

The least squares estimator of \( \theta \) is then

\[
\hat{\theta} = B^{-1}(\xi)\int_S z(x)y(x)\,d\xi(x),
\]

with bias vector and covariance matrix

\[
E[\hat{\theta} - \theta] = B^{-1}(\xi)b(f, \xi), \quad \text{cov}[\hat{\theta}] = \frac{\sigma^2}{n}B^{-1}(\xi).
\]

When there is no possibility of confusion we omit the dependencies of \( B, b \) and \( z \) on \( (f, \xi) \) and \( x \).

The mean squared error matrix is

\[
M(f, \xi) = E[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T] = \frac{\sigma^2}{n}B^{-1} + B^{-1}bb^TB^{-1}.
\]

Note that, with

\[
v = \sigma^2/n\eta^2,
\]

we can write

\[
\eta^{-2}M(\eta f, \xi) = (vB^{-1} + B^{-1}b(f, \xi)b^T(f, \xi)B^{-1}). \tag{2.1}
\]

The parameter \( v \) may then be interpreted as expressing the relative importance of variance versus bias, in the mind of the experimenter, in the neighbourhood \( ||f||_2 \leq 1 \).

We consider loss functions \( \mathcal{L}(f, \xi) \) satisfying

(L1) Monotonicity: If \( M(f_1, \xi) \geq M(f_2, \xi) \), in the sense of positive semi-definiteness, then \( \mathcal{L}(f_1, \xi) \geq \mathcal{L}(f_2, \xi) \);

(L2) Unboundedness: \( \mathcal{L}(f_n, \xi) \to \infty \) if \( \chi_1(M(f_n, \xi)) \), the maximum characteristic root of \( M(f_n, \xi) \), tends to infinity as \( n \to \infty \).

In practice, \( \xi \) must be discrete, with jumps consisting of integral multiples of \( n^{-1} \). We now drop this restriction. We search for a minimax design \( \xi \), i.e. one for which

\[
\sup_{f} \mathcal{L}(f, \xi) = \inf_{f} \sup \mathcal{L}(f, \xi),
\]

where the inf is taken over the class of all probability measures on \( S \).

We first fix a design \( \xi \), and maximize \( \mathcal{L}(f, \xi) \) over \( \mathcal{F} \). By Lemma 1 below, we may assume that \( \xi \) is absolutely continuous. Before proving this, we make the assumption, to avoid trivialities, that if there is a point \( x_0 \in S \) with \( z(x_0) = 0 \), then \( \xi\{x_0\} = 0 \). Otherwise, since such a point \( x_0 \) would contribute nothing to \( b \) or to \( B \), we could remove it from \( S \) and work with the conditional design on \( S \setminus \{x_0\} \).
**Lemma 1.** In order that \( \sup_{\varphi} \mathcal{L}(f, \xi) \) be finite, it is necessary that \( \xi \) be absolutely continuous.

**Proof.** Since \( M(f, \xi) \geq B^{-1}bb^T B^{-1} \), we have

\[
ch_1(M(f, \xi)) \geq ch_1(B^{-1}bb^T B^{-1}) = b^T B^{-2} b \geq \|b(f, \xi)\|^2 \cdot ch_B(B^{-2}).
\]

Here, \( \| \cdot \| \) is the Euclidean norm. Using (L2), it now suffices to show that

\[
\sup_{\varphi} \|b(f, \xi)\| = \infty
\]

(2.3)

if \( \xi \) is not absolutely continuous. But in this case, there is a set \( A \subset S \) with

\[
\int_A dx = 0 < \xi(A).
\]

By the assumption made preceding this lemma, we can choose \( f \in \mathcal{F} \) with \( \int_A zf(x) d\xi(x) \neq 0 \). Define, for real \( c \),

\[
f_c(x) = \begin{cases} f(x) & \text{if } x \in S \setminus A, \\ (c+1)f(x) & \text{if } x \in A. \end{cases}
\]

Then \( f = f_c \) a.e. (Lebesgue), so that \( f_c \in \mathcal{F} \). However,

\[
b(f_c, \xi) = b(f, \xi) + c \int_A zf(x) d\xi(x).
\]

so that

\[
\|b(f_c, \xi)\| \geq |c| \cdot \left\| \int_A zf(x) d\xi(x) \right\| - \|b(f, \xi)\|.
\]

Now (2.3) follows, upon letting \( c \to \infty \). □

By Lemma 2, we may restrict attention to those \( f \in \mathcal{F} \) with \( \|f\|_2 = \eta \).

**Lemma 2.** If \( f \in \mathcal{F} \) has \( \|f\|_2 = \varepsilon \eta < \eta \), then \( g = \varepsilon^{-1} f \in \mathcal{F} \), with \( \|g\|_2 = \eta \) and \( \mathcal{L}(g, \xi) \geq \mathcal{L}(f, \xi) \).

**Proof.** Note that

\[
M(g, \xi) - M(f, \xi) = (\varepsilon^{-2} - 1)B^{-1}(\xi)b(f, \xi)b^T(f, \xi)B^{-1}(\xi) \geq 0,
\]

and apply (L1). □

Now let \( m(x) = \xi'(x) \) be the density of \( \xi \). Define \( p \times p \) matrices

\[
A = \int_S z(x)z^T(x) dx, \quad C = \int_S z(x)z^T(x)m^2(x) dx, \quad G = C - BA^{-1}B.
\]
Note that
\[ G = \int_S [(m(x)I - BA^{-1})z(x)][(m(x)I - BA^{-1})z(x)]^T \, dx, \]
so that $G$ is positive definite. Define
\[ r(x) = \eta G^{-1/2}(m(x)I - BA^{-1})z(x), \]
so that
\[ \int_S r(x)r^T(x) \, dx = \eta^2 I, \]
\[ \int_S z(x)r^T(x) \, dx = 0, \]
\[ \int_S f(x)r(x) \, dx = \eta G^{-1/2} b(f, \xi), \]
\[ \int_S z(x)r^T(x) \, d\xi(x) = \eta G^{1/2}. \]

The following theorem reduces the search for a maximizing $f$ to a finite dimensional search in $\mathbb{R}^p$.

**Theorem 1.** Define
\[ \mathcal{H} = \{ h_\beta(x) = r^T(x)\beta; \| \beta \| = 1 \}. \]
Then:
(i) $\mathcal{H} \subset \mathcal{F}$, with $\| h \|_2 = \eta$ for all $h \in \mathcal{H}$;
(ii) $\sup_{h \in \mathcal{H}} \mathcal{L}(h, \xi) = \sup_{\| \beta \| = 1} \mathcal{L}(h_\beta, \xi)$.

**Proof.** Assertion (i) follows from (2.5a), (2.5b). For (ii), let $f \in \mathcal{F}$ be arbitrary. We will exhibit an $h \in \mathcal{H}$ with $b(h, \xi) = c b(f, \xi)$, $c \geq 1$. As in the proof of Lemma 2 it will then follow that $\mathcal{L}(h, \xi) \geq \mathcal{L}(f, \xi)$.

For any $h \in \mathcal{H}$, (2.5c) and the Cauchy-Schwarz inequality give
\[ \eta^2 \geq \left| \int_S f(x)h_\beta(x) \, dx \right| = \eta \| \beta^T G^{-1/2} b(f, \xi) \| . \]

Put $\beta_f = G^{-1/2} b(f, \xi)/\| G^{-1/2} b(f, \xi) \|$. Then with $\beta = \beta_f$, $h_\beta(x)$ has, by (2.5d),
\[ b(h, \xi) = \eta G^{1/2} \beta_f = c b(f, \xi). \]

By (2.6),
\[ c = \eta/\| G^{-1/2} b(f, \xi) \| \geq \| \beta_f^T G^{-1/2} b(f, \xi) \|/\| G^{-1/2} b(f, \xi) \| = 1. \]

**Example 1.** $\mathcal{L}(f, \xi) = |M(f, \xi)|$. This is maximized, for absolutely continuous $\xi$, by maximizing
\[ \mathcal{L}(h_\beta, \xi) = (\sigma^2/n)\eta^2(1 + \nu^{-1} \beta^T G^{1/2} B^{-1} G^{1/2} \beta) / |B|. \]
over $||\beta|| = 1$. For this, first determine $ch_1(G^{1/2}B^{-1}G^{1/2}) = \mu_\xi$, say. Let $\beta_\xi$ be the corresponding, normalized characteristic vector. Then

$$\sup_{\phi} D(f, \xi) = D(h_{\beta_\xi}, \xi) = (\sigma^2/n)^{\nu}(1 + (\mu_\xi/\nu))/|B|.$$ 

**Example 2.** $D(f, \xi) = tr M(f, \xi)$. As in Example 1, we consider

$$D(h_{\beta_\xi}, \xi) = \eta^2(\nu tr B^{-1} + \beta^T G^{1/2}B^{-2}G^{1/2} \beta).$$

With $\mu_\xi = ch_1(G^{1/2}B^{-2}G^{1/2})$, and $\beta_\xi$ the corresponding normalized characteristic vector,

$$\sup_{\phi} D(f, \xi) = D(h_{\beta_\xi}, \xi) = \eta^2(\nu tr B^{-1} + \mu_\xi).$$

### 3. Fitting a plane

In this section we construct minimax designs for the response function (1.3). As design space we take a sphere, in $\mathbb{R}^q$, of unit volume:

$$S = \{x \mid ||x|| \geq r := (\Gamma(\frac{1}{2}q + 1))^{1/q}/\sqrt{n}\}.$$ 

See Box and Draper (1959) for a discussion of this point.

Consider the group

$$\Pi = \{\pi : x \rightarrow \pi(x) = Q_\pi x, Q_\pi \text{ an orthogonal matrix}\}$$

of orthogonal transformations of $S$. For $f \in \mathcal{F}$ put $f_\pi(x) = f(\pi(x))$. Define $\mathcal{F}_\pi = \{f_\pi \mid f \in \mathcal{F}\}$. Then

$$\pi S = S, \quad \mathcal{F}_\pi = \mathcal{F}, \quad \text{for all } \pi \in \Pi.$$ 

Motivated by this, and the discussion in Kiefer (1959), we restrict to the class of $\Pi$-invariant designs, i.e. those which satisfy

$$D(f_\pi, \xi) = D(f, \xi) \quad \text{for all } \pi \in \Pi \text{ and } f \in \mathcal{F}.$$ 

Within this class we seek minimax designs. Let

$$d(x; f, \xi) = z^T(x)M(f, \xi)z(x),$$

the MSE of $z^T\theta$ as estimator of $z^T\theta$. Consider the loss functions

$$D_D(f, \xi) = |M(f, \xi)|, \quad D_A(f, \xi) = tr M(f, \xi),$$

$$D_E(f, \xi) = ch_1(M(f, \xi)), \quad D_Q(f, \xi) = \int_S d(x; f, \xi) \, dx,$$

$$D_G(f, \xi) = \sup_{x \in S} d(x; f, \xi).$$
These loss functions correspond to the classical notions of D-, A-, E-, Q- and G-optimality if $\eta = 0$. For them, the class of admissible, $\Pi$-invariant designs, henceforth denoted by $\mathcal{Z}_\Pi$, coincides with the class of absolutely continuous, spherically symmetric probability measures on $S$. A realization of a randomized design could be obtained by first drawing $n$ values $u_i$ of $\|X\|$ from a density as at (3.1) below, and then choosing $x_i$ from the uniform distribution on $\|X\| = u_i$, for each $i$.

In the case of $\mathcal{Z}_D$, it is a consequence of Lemmas 2.1 and 2.2 of Notz (1989) that the optimal design in $\mathcal{Z}_\Pi$ is in fact optimal in the class of all probability measures on $S$. (Notz makes an additional claim, applicable to our $\mathcal{Z}_A$. Unfortunately, this claim relies on an erroneous factorization of the matrix $N_j$ at p. 47 of Notz (1989).)

Any measure $\xi$ in $\mathcal{Z}_\Pi$ has a density which depends on $x$ only through $|x|$: 

$$\xi'(x) = m(x) = g(|x|),$$

where

$$\int_0^r \frac{qu^{q-1}}{r^q} g(u) du = 1. \quad (3.1)$$

The integrand in (3.1) is the density of $|X| := U$. Define $\gamma = E[X_j^2]$, i.e.

$$\gamma = \int_0^r \frac{u^{q+1}}{r^q} g(u) du. \quad (3.2)$$

A special, limiting role is played by the uniform measure ($m(x) \equiv 1$), for which

$$\gamma_0 := E[X_j^2 \mid m(x) \equiv 1] = \frac{r^2}{q+2}.$$

In terms of

$$J_0(g; \gamma) = E[g(U)] - 1 = \int_0^r \frac{qu^{q-1}}{r^q} (g(u) - 1)^2 du,$$

$$J_1(g; \gamma) = E[U^2 g(U)/q] - (\gamma^2/\gamma_0) = \int_0^r \frac{u^{q+1}}{r^q} \left( g(u) - \frac{\gamma}{\gamma_0} \right)^2 du,$$

we find, from (2.4), that

$$r^T(x) = \eta \left( g(|x|) - 1 / \sqrt{J_0(g; \gamma)} \right) \left( g(|x|) - \frac{\gamma}{\gamma_0} \right) x^T / \sqrt{J_1(g; \gamma)}. \quad (3.3)$$

To apply Theorem 1, partition $\beta$ as $\beta = (\beta_0, \beta_1)^T$. Then

$$M(h_\beta, \xi) = \eta^2 \left[ \sqrt{J_0 J_1} \beta_0 \beta_1^T / \gamma \right]. \quad (3.4)$$

Put $\alpha = |\beta_1|^2 = 1 - \beta_0^2$. The characteristic polynomial is then given by

$$|\eta^{-2} M(h_\beta, \xi) - \lambda I| = \left( \frac{\gamma}{\gamma_0} \right)^{\nu - 2} \psi(\lambda; (1 - \alpha) J_0, \alpha J_1),$$
where
\[ \psi(\lambda; s, t) = \lambda^2 - \left[ \frac{1}{\gamma} \left( 1 + \frac{1}{\gamma} \right) s + \frac{t}{\gamma^2} \right] \lambda + \frac{1}{\gamma} \left[ s + \frac{t}{\gamma} \right]. \]

Note that \( \psi(v/\gamma; (1 - \alpha)J_0, \alpha J_1) \leq 0 \), so that
\[ c_h(M(h, \xi)) = \eta^2 \lambda_1(\alpha; g), \quad (3.5) \]
where \( \lambda_1(\alpha; g) \) is the largest zero of \( \psi(\lambda; (1 - \alpha)J_0, \alpha J_1) \).

The proof of Theorem 2 below relies on the following lemma.

**Lemma 3.** Let \( a(a) = a + a'a, b(a) = b + b'a \) be linear functions of \( a \in [0, 1] \). Suppose that \( \psi(\lambda) = \lambda^2 - a(\alpha)\lambda + b(\alpha) \) has two real roots \( \lambda_1(\alpha) \geq \lambda_2(\alpha) \) for each \( \alpha \in [0, 1] \). Then \( \lambda_1(\alpha) \) is a monotonic function of \( \alpha \).

**Proof.** Put \( p(\alpha) = a^2(\alpha) - 4b(\alpha) \), so that \( \lambda_1(\alpha) = \frac{1}{2}(a(\alpha) + \sqrt{p(\alpha)}) \) and the quadratic \( p(\alpha) \) is non-negative in \([0, 1]\). If \( p(\alpha) \) has a zero \( \alpha_0 \in (0, 1) \), then \( p(\alpha) = a^2(\alpha - \alpha_0)^2 \), and we calculate that then \( \lambda_1(\alpha) \) is constant on one side of \( \alpha_0 \), linear on the other, hence monotonic.

If \( p(\alpha) > 0 \) on \((0, 1)\), then \( \lambda_1(\alpha) > \lambda_2(\alpha) \) there, and \( \lambda_1(\alpha) \) is differentiable, with derivative
\[ \lambda_1'(\alpha) = \frac{a'(\alpha) \lambda_1(\alpha) - b'}{\lambda_1(\alpha) - \lambda_2(\alpha)}. \]

If \( \lambda_1(\alpha) \) has no critical points in \((0, 1)\), then it is monotone on \([0, 1]\). Suppose, then, that \( \lambda_1(\alpha) \) has a critical point \( \alpha_* \in (0, 1) \). Then \( a' \neq 0 \), \( \lambda_1(\alpha_*) = b'/a' \), and \( \psi(\lambda_1(\alpha_*)) = 0 \). From this last equality we may now solve for \( b \) in terms of \( a, a', b' \). We find
\[ p(\alpha) = \left( a' + \left( a - \frac{2b'}{a'} \right) \right)^2, \]
whence
\[ \lambda_1(\alpha) = \max \left( \frac{b'}{a'}, \frac{a' + a - \frac{b'}{a'}}{a'} \right). \]

The point at which \( \lambda_1(\alpha) \) changes slope is the zero of \( p(\alpha) \), which by assumption lies outside of \((0, 1)\). Thus \( \lambda_1(\alpha) \) is either constant or linear in \([0, 1]\), hence is monotonic. \( \square \)

Let \( \mathcal{G} \) be the class of non-negative functions \( g, \) on \([0, r]\), satisfying \((3.1)\). For \( g \in \mathcal{G} \) and \( \alpha \in [0, 1] \) define:
\[ J_0(\alpha, g) = \eta^2 \left( \frac{v}{\gamma} \right)^q \left[ v + (1 - \alpha)J_0(g; \gamma) + \frac{\alpha J_1(g; \gamma)}{\gamma} \right], \quad (3.6) \]
\[ J_\xi(\alpha, g) = \eta^2 \left[ v + \frac{qv}{\gamma} + (1 - \alpha)J_0(g; \gamma) + \frac{\alpha J_1(g; \gamma)}{\gamma^2} \right], \quad (3.7) \]
\( \ell_E(\alpha; g) = \eta^2 \lambda_1(\alpha; g), \) \hspace{1cm} (3.8)

\( \ell_Q(\alpha; g) = \eta^2 \left[ 1 + \frac{q_0 \gamma_0}{\gamma} + (1 - \alpha) J_0(g; \gamma) + \frac{\alpha \gamma_0}{\gamma^2} J_1(g; \gamma) \right], \) \hspace{1cm} (3.9)

\( \ell_G(\alpha; g) = \eta^2 \left[ v + \frac{r^2}{\gamma} + \left( \sqrt{1 - \alpha} \sqrt{J_0(g; \gamma)} + \frac{r}{\gamma} \sqrt{\alpha \sqrt{J_1(g; \gamma)}} \right)^2 \right]. \) \hspace{1cm} (3.10)

**Theorem 2.** For \( \mathcal{L} \in \{ \mathcal{L}_D, \mathcal{L}_A, \mathcal{L}_E, \mathcal{L}_O, \mathcal{L}_G \} \), and \( \xi \in \mathcal{X}_P \), we have

\( \mathcal{D}(h_\beta, \xi) = \ell(\alpha; g). \) \hspace{1cm} (3.11)

For \( \mathcal{L}_D, \mathcal{L}_A, \mathcal{L}_E, \mathcal{L}_Q \):

\[ \inf_{\mathcal{L}} \sup_f \mathcal{D}(f, \xi) = \sup_g \max(\ell(0; g), \ell(1; g)). \] \hspace{1cm} (3.12)

For \( \mathcal{L}_G \):

\[ \inf_{\mathcal{L}} \sup_f \mathcal{D}_G(f, \xi) = \sup_g \ell_G(\alpha_*, g), \] \hspace{1cm} (3.13)

\[ \alpha_* = \left( 1 + \frac{\gamma^2 J_0(g; \gamma)}{r^2 J_1(g; \gamma)} \right)^{-1}, \] \hspace{1cm} (3.14)

\[ \ell_G(\alpha_*, g) = \eta^2 \left[ v + J_0(g; \gamma) + \frac{r^2}{\gamma} \left( v + \frac{J_1(g; \gamma)}{\gamma} \right) \right]. \] \hspace{1cm} (3.15)

**Proof.** We first verify (3.11). For \( \mathcal{L}_D, \mathcal{L}_A \), and \( \mathcal{L}_Q \), this follows easily from (3.4). For \( \mathcal{L}_E \) use (3.5). For \( \mathcal{L}_G \), first calculate

\[ d(x_0; h_\beta, \xi) = \eta^2 \left[ v + (1 - \alpha) J_0(g; \gamma) \right. \]

\[ + \frac{v}{\gamma} \left( x^T x + \frac{J_1(g; \gamma)}{\gamma} \right) (\beta_1^T x)^2 + 2 \left( \frac{J_0(g; \gamma) J_1(g; \gamma)}{v} \right)^{1/2} \beta_0 \beta_1^T x \].

The three summands within the braces are maximized simultaneously, over \( S \), at

\[ x_0 = r \cdot \text{sign}(\beta_0) \cdot \beta_1 \sqrt{\alpha}. \]

The expression at (3.10) is then \( d(x_0; h_\beta, \xi) \).

By virtue of Theorem 1,

\[ \inf_{\mathcal{L}} \sup_f \mathcal{D}(f, \xi) = \inf_{\mathcal{L}} \max_g \ell(\alpha; g). \]

Now (3.12) follows from the fact that \( \ell(\alpha; g) \) is monotonic in \( \alpha \), in each of the four cases. For \( \ell_E \), Lemma 3 is required here. A straightforward maximization of (3.10) gives (3.13)–(3.15). \( \Box \)

A general method of solving problems (3.12) is as follows. Represent \( \mathcal{G} \) as \( \mathcal{G}_0 \cup \mathcal{G}_1 \), where

\[ \mathcal{G}_k = \{ g \mid \ell(k; g) \geq \ell(1-k; g) \}. \]
Let $g_k(u)$ solve the problem:

$$
\text{Minimize } \ell(k; g), \text{ subject to } g \in \mathcal{G}_k.
$$

Put

$$
(\alpha_*, g_*) = \begin{cases} 
(0, g_0) & \text{if } \ell(0; g_0) \leq \ell(1; g_1), \\
(1, g_1) & \text{if } \ell(0; g_0) > \ell(1; g_1).
\end{cases}
$$

Then for any $g \in \mathcal{G}$,

$$
\max_{\alpha} \ell(\alpha; g_*) = \ell(\alpha_*; g_*) \leq \max_{\alpha} \ell(\alpha; g),
$$

so that $g_*$ is the desired minimax solution.

Suppose instead that $g_k$ solves the unconditioned problem:

$$
\text{Minimize } \ell(k; g), \text{ subject to } g \in \mathcal{G}.
$$

If it then turns out that for one value of $k$,

$$
g_k \in \mathcal{G}_k,
$$

then for all $\alpha$ and all $g$ we have

$$
\ell(\alpha; g_k) \leq \ell(k; g_k) \leq \ell(k; g).
$$

In this case, $(\alpha_*, g_*) = (k, g_k)$ gives a saddlepoint solution to problem (3.12).

We concentrate first on problem (3.13), and on those cases in which there exist saddlepoint solutions to (3.12). In these cases, as at (3.19), we first minimize $\ell(k; g)$ over $g$. We do this for fixed $y$. Equivalently, we minimize $(1 - \alpha)J_0 + \alpha J_1$, for either $\alpha = r^2/(y^2 + r^2)$, in (3.15), or $\alpha \in \{0, 1\}$, in (3.12). We then minimize $\ell(\alpha; g)$ over $y$.

Finally, in the case of problems (3.12), we check (3.20).

**Lemma 4.** The functional $(1 - \alpha)J_0(g; y) + \alpha J_1(g; y)$ is minimized over all $g$ satisfying (3.1), (3.2) by

$$
g_\alpha(u; y) = \frac{a(u^2 - br^2)^+}{(1 - \alpha)q + au^2}, \quad a > 0, \quad b \leq 1, \quad 0 \leq u \leq r.
$$

The constants $a, b$ are determined by (3.1), (3.2). (There are solutions corresponding to $a < 0$ as well, but then $y < y_0$, which is useless to us.)

**Proof.** It suffices for $g_\alpha(u; y)$ to minimize

$$
(1 - \alpha)J_0(g; y) + \alpha J_1(g; y) + 2abr^2 \int_0^r \frac{u^q - g(u)}{r^q} \, du - 2a \int_0^r \frac{u^{q+1}}{r^q} \, g(u) \, du
$$

$$
= \int_0^r \frac{u^{q-1}}{r^q} \left\{[(1 - \alpha)q + au^2]g^2(u) - [2a(u^2 - br^2)]g(u) - q \left(1 + \frac{y^2}{y_0}\right)\right\} \, du,
$$

for some Lagrange multipliers $a, b$, and to satisfy (3.1), (3.2). The $g_\alpha(u; y)$ of (3.22) minimizes the integrand of (3.23) pointwise. □
We require the details of \( g_\alpha(u;\gamma) \) in three cases. It is convenient to parametrize the solutions by \( \gamma/\gamma_0 \), which varies from 1, for the uniform design, to \((q + 2)/q\), for the degenerate design with all mass at \(|x| = r\). These will be seen to correspond to \( \nu = 0 \) and \( \nu = \infty \), respectively.

**Case 1: \( \alpha = 0 \).**

(a) We have \( b \leq 0 \) for

\[
1 \leq \frac{\gamma}{\gamma_0} \leq \frac{(q + 2)^2}{q(q + 4)}
\]

Then

\[
g_0(u;\gamma) = 1 + \left( \frac{\gamma}{\gamma_0} - 1 \right) \left( \frac{q + 4}{4} \right) \left( \frac{u^2}{\gamma_0} - q \right), \quad 0 \leq u \leq r,
\]

with

\[
J_0(g_0;\gamma) = \frac{q(q + 4)}{4} \left( \frac{\gamma}{\gamma_0} - 1 \right)^2, \quad J_1(g_0;\gamma) = \frac{(q + 2)^2\gamma_0}{4(q + 6)} \left( \frac{\gamma}{\gamma_0} - 1 \right)^2.
\]

(b) The range \( 0 \leq b \leq 1 \) corresponds to

\[
\frac{(q + 2)^2}{q(q + 4)} \leq \frac{\gamma}{\gamma_0} \leq \frac{q + 2}{q}.
\]

In terms of

\[
K_q(b) = (1 - b) - \frac{2(1 - b^{q/2 + 1})}{q + 2} = q \left[ \int_{\sqrt{b}}^{1} v^{q-1}(v^2 - b) \, dv \right],
\]

we find

\[
g_0(u;\gamma) = \left[ \left( \frac{u}{r} \right)^2 - b \right] / K_q(b), \quad r\sqrt{b} \leq u \leq r,
\]

\[
\frac{\gamma}{\gamma_0} = \frac{K_{q+2}(b)}{K_q(b)}, \quad J_0(g_0;\gamma) = \frac{q\gamma - br^2}{r^2 K_q(b)} - 1,
\]

\[
J_1(g_0;\gamma) = \left( \frac{\gamma_0}{K^2_q(b)} \right) \left( \frac{q + 2}{q + 4} \right) K_{q+4}(b) - (b + K_{q+2}(b)) K_{q+2}(b).
\]

**Case 2: \( \alpha = 1 \).** For \( 0 \leq b \leq 1 \), i.e.

\[
1 \leq \frac{\gamma}{\gamma_0} \leq \frac{q + 2}{q},
\]

we have

\[
g_1(u;\gamma) = a \left( 1 - b \left( \frac{r}{u} \right)^2 \right), \quad r\sqrt{b} \leq u \leq r,
\]

where

\[
a = \frac{q - 2}{q K_{q-2}(b)}, \quad \gamma = \frac{ar^2 K_q(b)}{q},
\]

and

\[
J_0(g_1;\gamma) = a - 1 - \frac{a^2 b q K_{q-4}(b)}{q - 4}, \quad J_1(g_1;\gamma) = a\gamma - \frac{abr^2}{q} - \frac{\gamma^2}{\gamma_0}.
\]
Case 3: \( \alpha = r^2/(\gamma^2 + r^2) \), \( b \leq 0 \).

Write \( g_* \) for the \( g \) minimizing (3.15). We give the details only for the case \( b \leq 0 \). In terms of

\[
S = \frac{r^2}{\gamma \sqrt{q} }, \quad C_j(s) = \int_0^{\tan^{-1}(s)} \tan^j \theta \, d\theta,
\]

we have

\[
g_*(u; \gamma) = \frac{as^2(u^2 - br^2)}{r^2(r^2 + s^2 u^2)}, \quad 0 \leq u \leq r, \tag{3.29}
\]

\[
a = \frac{r^2 s^q}{q} \left[ \frac{s\sqrt{q}C_{q-1}(s) - C_{q+1}(s)}{C_{q-1}(s)C_{q+1}(s) - C_{q+2}(s)} \right], \quad b = \frac{1}{s^2} \left[ \frac{s\sqrt{q}C_{q+1}(s) - C_{q+3}(s)}{s\sqrt{q}C_{q-1}(s) - C_{q+1}(s)} \right].
\]

Then \( a \geq 0 \geq b \) for \( s \in [s_0, (q + 2)/\sqrt{q}] \), i.e.

\[
1 \leq \frac{\gamma}{\gamma_0} \leq \frac{q + 2}{s_0 \sqrt{q}},
\]

where \( s_0 \) is the solution, in \((\sqrt{q}, (q + 2)/\sqrt{q})\), to

\[
s_0 \sqrt{q} - \frac{C_{q+3}(s_0)}{C_{q+1}(s_0)}.
\]

3.1. D-optimal designs

We claim that if \( \gamma \) is chosen to minimize

\[
I_D(0; g_0) = \eta^2 \rho \left( \frac{\gamma}{\gamma_0} \right)^q (\gamma + J_0(g_0; \gamma)), \tag{3.30}
\]

then (3.20) holds with \( k = 0 \), i.e.

\[
J_1(g_0; \gamma)/\gamma < J_0(g_0; \gamma). \tag{3.31}
\]

Then, as at (3.21), \((0, g_0)\) is the desired saddlepoint and

\[
m_D(x) = g_0(\|x\|; \gamma) \tag{3.32}
\]

is the density of the minimax design. The least favourable member of \( \mathcal{F} \) is \( f_0(x) = r^T(x) \beta \), with \( \beta = (1, 0^T) \). From (3.3),

\[
f_0(x) = \frac{\eta(g_0(\|x\|; \gamma) - 1)}{\sqrt{J_0(g_0; \gamma)}}. \tag{3.33}
\]

(a) Large values of \( n \) \( (b \leq 0) \). With (3.25) in (3.30), we find that \( I_D(0; g_0) \) is minimized by that unique \( \gamma \) satisfying

\[
\gamma = \left( \frac{q + 4}{2} \right) \left( \frac{\gamma}{\gamma_0} - 1 \right) \left( \frac{\gamma}{\gamma_0} - \frac{q}{2} \left( \frac{\gamma}{\gamma_0} - 1 \right) \right). \tag{3.34}
\]

Any \( \gamma \) satisfying (3.24) also satisfies (3.31). From (3.24) and (3.34) we then find that
this solution is valid for
\[ 0 \leq v \leq \frac{2(q^2 + 2q + 4)}{q^2(q + 4)}. \]

(b) Small values of \( n (b \geq 0) \). We consider the right side of (3.30) as a function of \( b \), through (3.26). Then (3.30) is minimized by
\[
v = \frac{2\gamma}{q} \left( \frac{1}{\gamma_0} \right) \left( 1 - \frac{b^{q/2}}{\gamma} \right) - \frac{2b^{q/2}}{\gamma_0}
\]
(3.35)

Relationship (3.31) holds for all \( b \in [0, 1] \). This corresponds, by (3.35), to
\[
\frac{2(q^2 + 2q + 4)}{q^2(q + 4)} \leq v < \infty.
\]

3.2. Q-optimal designs

These were constructed by Huber (1975) for \( q = 1 \), and Wiens (1990) for \( q > 1 \), under the assumption that \( \xi \) be absolutely continuous. The results in Section 2 of this paper then establish the overall optimality of such designs. For \( q \geq 1 \) they satisfy (3.32) and (3.33), as for D-optimal designs, but (3.34) and (3.35) are replaced by
\[
\gamma + 2 \gamma_0 \gamma \gamma_0
\]
respectively. The first form \( (b \geq 0) \) holds for
\[
0 \leq v \leq \frac{2(q + 2)^4}{q^3(q + 4)^2},
\]
while the second holds for all larger values of \( v \).

3.3. A-optimal designs, small \( n \)

We first minimize
\[
\ell_A(1; g_i) = \eta^2 \left( v + \frac{qv}{\gamma} + \frac{J_1(g_i; \gamma)}{\gamma^2} \right),
\]
viewed as a function of \( b \) via (3.27) and (3.28). We find that then
\[
v = \frac{2b}{qK_q(b)}.
\]
(3.37)

In this case (3.20), with \( k = 1 \), becomes
\[
J_1(g_i; \gamma)/\gamma^2 \geq J_0(g_i; \gamma).
\]
(3.38)
This turns out to hold only for a range $b \in [b_A, 1]$, where $b_A$ satisfies (3.38) with equality. Correspondingly, $v \geq v_A$, given by (3.37) with $b = b_A$. For $q = 1$ (simple linear regression) we find $v_A = \frac{1}{4}$. For $q = 2$, $v_A = 0.0671$.

For $v \geq v_A$ the problem then has as saddlepoint solution the pair
\[
(f_A(x), m_A(x)) = \left( \frac{\eta(g_1(|x|; \gamma) - \gamma/\gamma_n)}{\sqrt{J_1(g_1; \gamma)}}, g_1(|x|; \gamma) \right),
\]
where $\gamma$ is given by (3.27) and (3.37) and where $\beta_1$ is any vector of unit length. For $v < v_A$ we have neither $g_0 \in \mathcal{G}_0$, nor $g_1 \in \mathcal{G}_1$. It follows that for this range there are no saddlepoint solutions to the $A$-optimality problem. Non-saddlepoint solutions are discussed in subsection 3.6, below.

3.4. $E$-optimal designs, small $n$

The situation here is very similar to the $A$-optimality case. Replace (3.36)-(3.38) by
\[
\zeta_E(1; g_1) = \eta^2 \left( \frac{v}{\gamma} + \frac{J_1(g_1; \gamma)}{\gamma^2} \right), \quad v = \frac{2b}{K_d(b)},
\]
\[
v + J_0(g_1; \gamma) \leq \frac{v}{\gamma} + \frac{J_1(g_1; \gamma)}{\gamma^2},
\]
respectively. The saddlepoint solutions are then as at (3.39) for $v \geq v_E$, obtained from (3.40) and (3.41) (with equality). For $q = 1$ we find $v_E = 0.00271$ while for $q = 2$, $v_E = 6.236 \times 10^{-14}$. Note that for $q = 1$ and $v \geq v_A$, the $A$- and $E$-optimal designs are identical.

Again, there are no saddlepoint solutions if $v < v_E$.

3.5. $G$-optimal designs

We give the details only for large values of $n$, i.e. $b \leq 0$. For $b > 0$ the expressions are considerably more complex. With (3.29) in (3.15), we find
\[
\zeta_G(\alpha_*, g_*) = \eta^2 \left[ v(1 + s\sqrt{q}) + \frac{as\sqrt{q}}{r^2} - \frac{abs^2}{r^2} - (q + 3) \right].
\]
This is minimized by that $s$ satisfying
\[
v = \frac{1}{\sqrt{q}} \left[ \frac{(q + 2 - s\sqrt{q})^2 C_{q+1}(s)}{qs^2 C_{q+1}(s) - (q + 2)C_{q+3}(s)} \right].
\]
For $s \in [s_0, (q + 2)/\sqrt{q}]$ the solution is then valid for
\[
0 \leq v \leq v(s_0) = \frac{(q + 2)^2}{q\sqrt{q}} \frac{(1 + s_0\sqrt{q})(s_0 - \sqrt{q})}{s_0^2(1 + s_0^2)}.
\]
For $q = 1$, $s_0 = 2.0164$, $v(s_0) = 1.3396$. For $q = 2$, $s_0 = 2.4689$, $v(s_0) = 0.6197$. 
The minimax design, for \( v \leq v(s_0) \), is given by

\[
m_G(x) = g_\ast(\|x\|; \gamma),
\]

with \( \gamma = r^2/(s\sqrt{q}) \), and \( s \) given by (3.42). The least favourable member of \( \mathcal{F} \) is

\[
f_G(x) = \frac{\eta}{\varepsilon} \left( g_\ast(\|x\|; \gamma) - 1 + r\beta_1^T x \left( g_\ast(\|x\|; \gamma) - \frac{\gamma}{\gamma_0} \right) \right)
\]

where

\[
\varepsilon^2 = r^2 J_1(g_\ast; \gamma) + \gamma J_0(g_\ast; \gamma) = a\gamma - \frac{abr^2}{q} - (q + 3)\gamma^2
\]

and \( \beta_1 \) is any vector of unit length.

### 3.6. A- and E-optimal designs, for large \( n \)

As at (3.16)-(3.18) it is possible to construct minimax A- and E-optimal designs, for large \( n \), which do not correspond to saddlepoints. The details are as follows.

First construct \( g_0(u; t, \gamma) \) to minimize \( J_0(g; \gamma) \), subject to

\[
J_1(g; \gamma) = t.
\]

As in Lemma 4, we find that

\[
g_0(u; t, \gamma) = \left( \frac{a + bu^2}{1 + cu^2} \right)^+ I(0 \leq u \leq r),
\]

with \( a, b, c \) satisfying (3.1), (3.2) and (3.43).

Similarly,

\[
g_1(u; s; \gamma) = \left( \frac{a' + b'u^2}{c' + u^2} \right)^+ I(0 \leq u \leq r)
\]

minimizes \( J_1(g; \gamma) \), subject to

\[
J_0(g; \gamma) = s,
\]

if \( a', b', c' \) satisfy (3.1), (3.2) and (3.44). For A-optimality, (3.16) then requires us to

1. Determine \((t_\ast, \gamma_\ast)\) to minimize \( \ell_A(0; g_0(\cdot; t, \gamma)) \), subject to \( g_0 \in \mathcal{G}_0 \), i.e.

\[
\frac{t}{\gamma} \leq J_0(g_0; \gamma).
\]

2. Determine \((s_\ast, \gamma_\ast)\) to minimize \( \ell_A(1; g_1(\cdot; s, \gamma)) \) subject to \( g_1 \in \mathcal{G}_1 \), i.e.

\[
s\gamma^2 \leq J_1(g_1; \gamma).
\]

The minimax design is then given by

\[
m(x) = \begin{cases} 
g_0(\|x\|; t_\ast, \gamma_\ast) & \text{if } \ell_A(0; g_0(\cdot; t_\ast, \gamma_\ast)) \leq \ell_A(1; g_1(\cdot; s_\ast, \gamma_\ast)), 
g_1(\|x\|; s_\ast, \gamma_\ast) & \text{otherwise.} 
\end{cases}
\]
Table 1
Constants for optimal designs; \( q = 1, r = \frac{1}{4}, \gamma_0 = \frac{1}{15} \)

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<th>( \nu )</th>
<th>0</th>
<th>0.01</th>
<th>0.40</th>
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<td>( \gamma_0 )</td>
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<td>0.0957</td>
<td>0.1118</td>
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<td>0.0938</td>
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<td>0.1413</td>
<td>0.1637</td>
<td>0.1985</td>
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<tr>
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<td>( b )</td>
<td>0.0938</td>
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<td>0.1413</td>
<td>0.1637</td>
<td>0.1985</td>
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<tr>
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<tr>
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<td>( a )</td>
<td>2.345</td>
<td>5.460</td>
<td>8.816</td>
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Table 2
Constants for optimal designs; \( q = 2, r = 1/\sqrt{\pi}, \gamma_0 = 1/4\pi \)

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<td>0.1061</td>
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<td>1/2( \pi )</td>
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<td>Q</td>
<td>( y )</td>
<td>( \gamma_0 )</td>
<td>0.0798</td>
<td>0.0882</td>
<td>0.0973</td>
<td>0.1226</td>
<td>0.1314</td>
<td>0.1440</td>
</tr>
<tr>
<td></td>
<td>( b )</td>
<td>0</td>
<td>0.0882</td>
<td>0.0973</td>
<td>0.1226</td>
<td>0.1314</td>
<td>0.1440</td>
<td>1/2( \pi )</td>
</tr>
<tr>
<td>A</td>
<td>( y )</td>
<td>( \gamma_0 )</td>
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<td>0.1125</td>
<td>0.1321</td>
<td>0.1387</td>
<td>0.1482</td>
<td>1/2( \pi )</td>
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<tr>
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<td>( a )</td>
<td>1.744</td>
<td>2.637</td>
<td>7.333</td>
<td>13.587</td>
<td>46.544</td>
<td>( \infty )</td>
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<tr>
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<td>( b )</td>
<td>0.1459</td>
<td>0.2679</td>
<td>0.5367</td>
<td>0.6417</td>
<td>0.8000</td>
<td>1</td>
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<tr>
<td>E</td>
<td>( y )</td>
<td>( \gamma_0 )</td>
<td>0.0806</td>
<td>0.0943</td>
<td>0.1038</td>
<td>0.1242</td>
<td>0.1321</td>
<td>0.1441</td>
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<td>( a )</td>
<td>1.018</td>
<td>1.412</td>
<td>1.901</td>
<td>4.641</td>
<td>7.733</td>
<td>24.808</td>
<td>( \infty )</td>
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<tr>
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<td>( b )</td>
<td>0.0025</td>
<td>0.0839</td>
<td>0.1716</td>
<td>0.4202</td>
<td>0.5367</td>
<td>0.7298</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>( y )</td>
<td>( \gamma_0 )</td>
<td>0.0798</td>
<td>0.0873</td>
<td>0.1061</td>
<td>0.1326</td>
<td>0.1394</td>
<td>0.1484</td>
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<tr>
<td>&amp;</td>
<td>( s )</td>
<td>( 2y/2 )</td>
<td>2.821</td>
<td>2.578</td>
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<tr>
<td>&amp;</td>
<td>( a )</td>
<td>0.3183</td>
<td>0.3207</td>
<td>0.4155</td>
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<tr>
<td>&amp;</td>
<td>( b )</td>
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</table>
For E-optimality, $\ell_A$, (3.45) and (3.46) are replaced by $\ell_E$,

$$v + J_0(g_0; \gamma) \geq \frac{v}{\gamma} + \frac{t}{\gamma^3}$$

and

$$v + s \leq \frac{v}{\gamma} + \frac{J_1(g_1; \gamma)}{\gamma^2},$$

respectively.

We do not work out the particulars here, since the solutions are probably too complicated for practical purposes.
3.7. Numerical comparisons

Some values of the constants, for the minimax designs of Sections 3.2 to 3.5, are presented in Table 1 \((q = 1)\) and Table 2 \((q = 2)\). It is seen there that for fixed \(v\), the D- and Q-optimal designs are almost indistinguishable, as are the A- and E-optimal designs. In Figure 1, graphs of the D-, A- and G-optimal densities are shown, for \((q, v) = (1, 1)\) and \((q, v) = (2, 0.4)\).

References