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Some Comments on Coordinate-Free and Scale-Invariant Methods in Morphometrics

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ABSTRACT The usual strategy for comparing biological shapes is to use some kind of superimposition of the two forms under study and then look at the "residuals" as the shape change. In this paper, I take a careful look at this general strategy and point out some subtle but inherent and important pitfalls. Additionally an alternative approach based on Euclidean Distance Matrix representation is presented. It is applicable to two- as well as three-dimensional objects.

One obvious manifestation of biological processes such as growth, evolution, or teratogenesis is change in the form of an object. Form of an object involves both size and shape. In order to quantitatively compare forms and shapes we need a method for cataloguing the forms under consideration. Two types of data that are commonly used for this procedure are landmark data and outline data. In this paper I consider analysis of landmark data, although many of the comments extend naturally to outline data.

Several different methods have been developed for comparing shapes using landmark data (e.g., Bookstein 1978, 1986; Siegel and Benson, 1982; Goodall and Bose, 1987; Lewis et al., 1980). The purpose of this paper is to take a careful look at these approaches and raise a few philosophical points with important practical implications. This paper also proposes a new method for comparing forms based on Euclidean Distance Matrix representation. The proposed method works for three-dimensional objects and gives biologically interpretable quantities.

In this discussion, the form of the object refers to the geometric representation of the object by the landmarks. The curvature and other features of the surfaces between landmarks which may contain important information about the form of an object are lost in the analysis of landmark data. The limitations of landmark data are recognized and accepted throughout this paper.

For the sake of simplicity of exposition, I consider only the case in which one is comparing two objects for which landmark data are available. Comparing forms or shapes of two groups of objects is considered in Lele and Richtsmeier (1991a,b). Throughout this paper, form of an object is defined to be that characteristic which remains invariant under translation, rotation, and reflection of the object. Shape is defined to be that characteristic which remains invariant under translation, rotation, reflection, and scaling.

SUPERIMPOSITION METHODS

With the exception of Finite Element Scaling Analysis (Lewis et al., 1980), almost all morphometric methods employ superimposition to calculate form or shape difference. Let us look closely at how superimposition is implemented for landmark data. Let X and Y be two $(K \times 2)$ or $(K \times 3)$ matrices of landmark coordinates where K is the number of landmarks. A general procedure for superimposition of these figures can be described as follows:

Step 1: Fix one of the figures, say X, as the reference figure.

Step 2: Select positive real valued functions $\phi_i(\cdot)$ called the loss functions, $i = 1,2,\ldots,K$. Let $d(i_X,i_Y)$ be the distance between landmark i in figure X and figure Y, respectively.

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Step 3: Translate and rotate figure Y so that $\sum_{i=1}^{K} \phi_i[d(i_X, i_Y)]$ is minimized.

Step 4: If one wants only the shape difference, scale the figure Y so that $\sum_{i=1}^{K} \phi_i[d(i_X, i_Y)]$ is minimized.

Following are two examples of the loss functions $\phi_i(\cdot)$.

1. Ordinary Procrustes Analysis (Goodall and Bose, 1987): In this procedure two figures are superimposed in such a manner that the sum of the squared distances between corresponding landmarks is minimized. Hence the corresponding loss function is given by

$$\phi_i(x) = x^2 \quad \text{for all } i$$

2. Weighted Ordinary Procrustes Analysis (Goodall, 1991): In this procedure one minimizes the weighted sum of squared distances between the corresponding landmarks. The corresponding loss function is given by

$$\phi_i(x) = W_i x^2 \quad \text{where } W_i \text{ are preselected weights}$$

There are infinitely many different functions that can be chosen as loss functions. As a result, by selectively choosing a loss function one can support almost any hypothesis about how two forms differ. This is demonstrated by considering two triangles X and Y with the following landmark coordinates:

$$X = \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Comparing these two triangles using superimposition schemes with different loss functions yields quite different results.

Edge matching method: In this method one fixes a particular edge, say (1,2) in object X (Fig. 1a) and then translates, rotates, and scales Y such that the edge (1,2) in Y matches with the same edge in X exactly. Because there are three different edges (1,2), (2,3), and (1,3), three different conclusions about where and how the two shapes differ can be drawn (Fig. 1a–c). Although the particulars vary, all changes appear to occur at only one landmark when using this method.

Ordinary Procrustes Analysis: Figure 1d shows the direction and magnitude of the shape difference as depicted by this method. It concludes that changes have occurred local to all three landmarks.

If one performs multiple weighted ordinary procrustes analyses with different weights, different shape changes can be produced.

Robust theta-rho fit (Siegel and Benson, 1982): Figure 1e shows the results of the robust fit algorithm. This method concludes that changes have occurred at two of the three landmarks.

Even when the same two objects are being compared, vastly different conclusions about how they differ in shape seem possible. Scientifically, these varying conclusions are unsettling.

In mathematical terms the problem with superimposition methods can be stated as follows. Following Goodall (1991), suppose $Y = b(X + J)B + 1_k t'$ where $b > 0$ is a scalar, B is an orthogonal matrix, and t is a vector. Here t corresponds to translation, b corresponds to size difference, B corresponds to rotation, and, finally, J corresponds to the "shape difference." But (b,t,B,J) are nonidentifiable! That is, there are many combinations of these four variables that can lead us from X to Y, as illustrated in Figure 1. Which combination should we take as the true one? To make the problem identifiable, superimposition methods use the following constraint: Choose (b,t,B,J) such that $\sum_{i=1}^{K} \phi_i[d(i_X,i_Y)]$ is minimized for arbitrarily chosen functions $\phi_i(\cdot)$. It is obvious that the choice of $\phi_i(\cdot)$ affects inferences about J, the shape difference. In my opinion, there is no convincing argument for choosing any particular loss function over others.

A biologist is not interested in merely testing the null hypothesis of similarity of forms or shapes, but also in localizing form/shape differences. For example, in the study of human dysmorphology when planning for corrective plastic surgeries, it is critical for the surgeon to know where and by how much the forms are different. In evolutionary studies it is important to know where morphological changes have occurred because the nature of such changes may impact on systematic, functional, or paleontological hypotheses. Because the choice of the loss function $\phi(\cdot)$ affects J, the shape difference matrix, localization of the differences in form is problematic when using superimposition to find the shape differences.

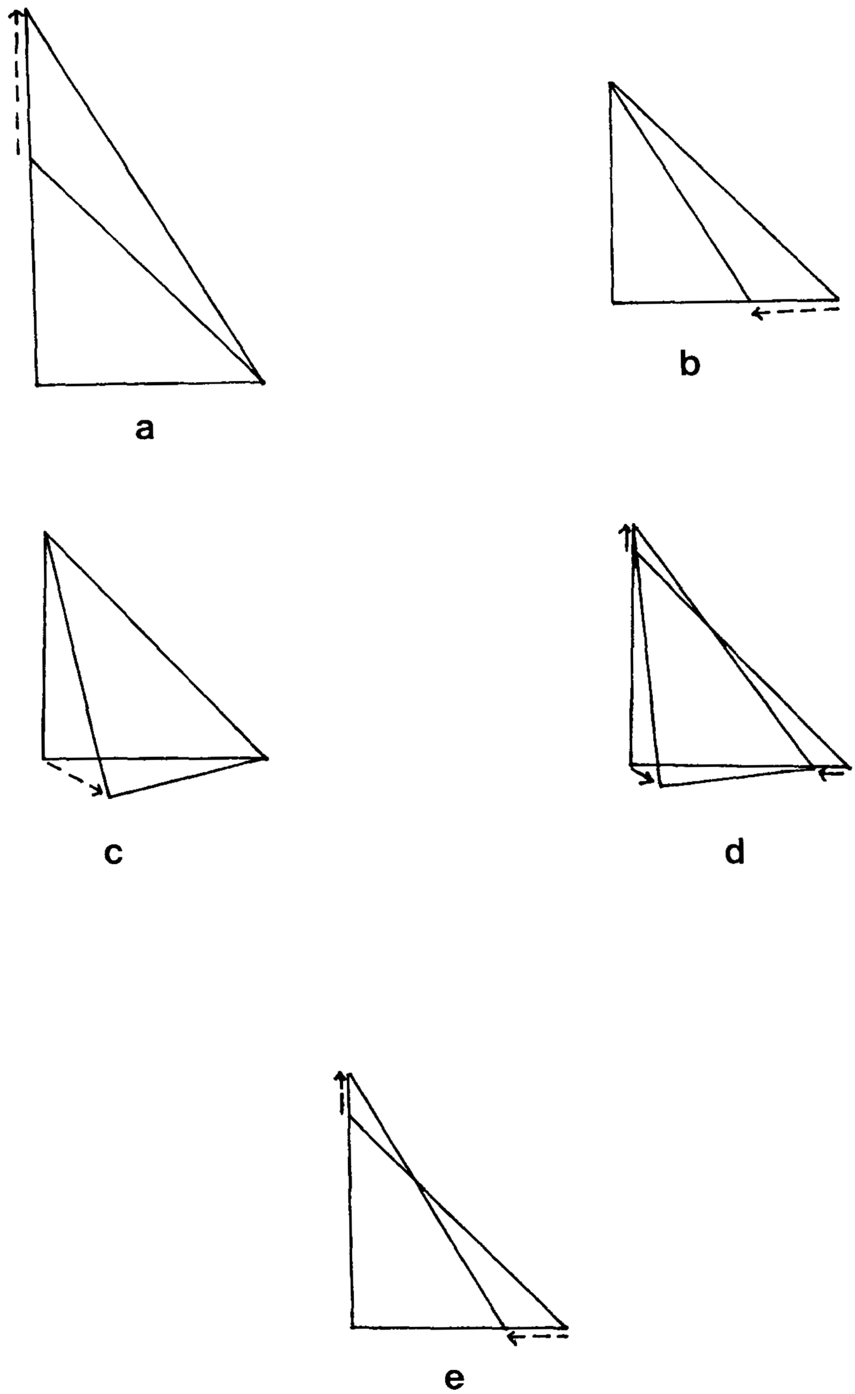

Fig. 1. Shape difference as shown by the dotted vectors between the same two triangles using different loss functions: (a–c) The conclusions drawn by method of edge matching. Changes are attributed to one landmark only, the landmark however depends on the matched edge which is chosen arbitrarily. (d) The conclusion drawn by the ordinary procrustes analysis. This says that all three landmarks have changed. (e) The conclusion drawn by the robust theta-rho fit. It says that only two landmarks have changed. This example thus illustrates the arbitrariness of the conclusions drawn by superimposition methods.

It should be noted that various methods for comparing shapes where only outline data are available can be looked upon as superimposition methods. All the above criticisms apply to them as well.

Is the situation hopeless? I do not think so. There are at least two methods for comparing biological shapes that do not involve superimposition.

AVOIDING SUPERIMPOSITION

Finite element scaling analysis (FESA) proposed by Lew and Lewis (1977) compares two forms without superimposition. A detailed discussion of this method is available in Cheverud and Richtsmeier (1986). Although the method does not rely on superimposition, in my opinion, the following features of this method are troublesome:

1. Choice and effect of the homology function: The homology function determines the plotting of the pseudohomologous points in the interior of the element. The choice of this function can alter the form difference.
2. Choice of the element shape and design: The type of elements used and how the object is discretised depend on the experimenter. Unfortunately both these choices affect the form difference (Richtsmeier et al., 1989).
3. When a landmark is shared by different elements, there seems to be no unique way to calculate the form difference at such landmarks.

Thus results of the form comparisons using superimposition methods are affected by the choice of the loss function whereas results from FESA are affected by the choice of the homology function and the element design. On the positive side, FESA can be used to represent the form difference graphically using Thompsonian-type grids. However, one should remember that these graphics very much depend on the homology function, which may not represent the physical properties of the interior.

Some of these concerns regarding superimposition methods and FESA have been raised previously in the literature. For a recent review see Lestrel (1989).

INVARIANCE PRINCIPLE, MAXIMAL INVARIANTS, AND COMPARISON OF FORMS

In this section, I discuss a principle which can be used to evaluate different methods of form comparison. Mathematically oriented readers may refer to Cox and Hinkely (1974) for more details on the invariance principle and maximal invariants. Here I discuss these ideas at a mathematically less rigorous level.

As defined earlier, the form of an object is that characteristic which remains invariant under translation, rotation, and reflection of the object. This definition suggests the following principle.

Invariance principle

All the scientific inferences concerning the forms of objects should remain invariant under translation, rotation, and reflection of the objects.

To illustrate the invariance principle, consider a two-dimensional object with four landmarks. This object is represented by a (4×2) matrix of real numbers consisting of (X, Y) coordinates of four landmarks. Now suppose we translate and rotate this object and measure the coordinates of the same four landmarks. The (4×2) matrix now obtained is different than the original (4×2) matrix. The following two matrices, although different, correspond to the same object:

$$X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad X^* = \begin{bmatrix} 1.5 & 2.0 \\ 1.5 & 3 \\ 0.5 & 3 \\ 0.5 & 2 \end{bmatrix}$$

In fact,

$$X^* = XB + \mathbf{1}\ \mathbf{t}$$

where

$$B = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

$$\mathbf{t} = \begin{pmatrix} 1.5 & 0 \\ 0 & 2.0 \end{pmatrix}$$

and

$$\mathbf{1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

In general, any translation, rotation and reflection of X can be expressed as

$$X^* = XB + \mathbf{1}\,\mathbf{t}$$

where B is a (2×2) orthogonal matrix corresponding to rotation and reflection, $\mathbf{t}$ is a 2×2 diagonal matrix of real numbers corresponding to translation, and $\mathbf{1}$ is a matrix of 1s. Similar operations can be defined for a three-dimensional object.

Note that if one is interested only in the form of the object, given that form of an object is invariant under translation, rotation, and reflection, the representations X and X^* are equivalent.

Let us fix X and consider the collection of all matrices X^*s that can be obtained by choosing different values of B and $\mathbf{t}$. All of these X^*s are equivalent to X. In fact every matrix in this collection is equivalent to every other matrix. Note also that X belongs to this collection when $\mathbf{t} = 0$ and $B = I$. We refer to such a collection of all matrices which are equivalent to each other (because they are translations, rotations, and/or reflections of each other) as an "orbit."

Consider the space of all $(K \times 2)$ matrices. This space corresponds to landmark coordinate matrices of two dimensional objects with K landmarks. An object can be considered to be a "point" in this space. All the "points" that lie on the same orbit are equivalent, and conversely if two "points" are equivalent then they lie on the same orbit (see Fig. 2). One can think of these orbits as the equithermals on a weather map or the contours on a topographical map.

Now let X and Y be two different objects in the sense that they are not equivalent to each other, i.e., they lie on two different orbits in the landmark coordinate space. How should we quantify the "form difference" between X and Y?

Suppose we simply take a coordinatewise difference between X and Y viz. $X - Y$ and define it as the form difference between X and Y. Clearly this "form difference" is not invariant under rotation or translation of X. That is, let $X^* = XB + \mathbf{1}\,\mathbf{t}$, then $X - Y$ is not equal to $X^* - Y$. But the invariance principle demands that the definition of form difference be such that it is invariant under translation, rotation, and reflection of X or Y or both. Hence this naive definition of form difference does not seem satisfactory.

To quantify the form difference between X and Y properly, we need to introduce the concept of maximal invariant. Let $M(\cdot)$ be a function defined on the landmark coordinate space such that it assigns the same value to all the points which are on the same orbit but assigns different values for points that are on different orbits. Thus, if X and Y are equivalent then $M(X) = M(Y)$, and if X and Y are not equivalent then $M(X) = M(Y)$. Such a function $M(\cdot)$ is called a *maximal invariant.* It is also important that this function $M(\cdot)$ be such that given its value one can construct the configuration of K points representing the original object, thus retaining all the information about the form of an object as represented by K landmarks.

Suppose such a function $M(\cdot)$ exists. Then the domain of this function is the landmark coordinate space and the range of this function is called a maximal invariant space. Note that an orbit in the landmark coordinate space maps to a single point in the maximal invariant space (see Fig. 2).

Suppose now we define form difference between X and Y in terms of $M(X)$ and $M(Y)$, then this form difference (whatever its definition is!) is invariant to the rotation, reflection, and translation of X or Y or both. This follows because $M(X)$ and $M(Y)$ are invariant to these operations, the form difference defined in their terms thus satisfies the invariance principle. This suggests that form difference should be defined and studied in the maximal invariant space.

EUCLIDEAN DISTANCE MATRIX ANALYSIS

Since the form of an object is invariant under translation, rotation, and reflection, it follows from the previous section that an approach for comparing forms should start with a representation which is invariant under these operations. Such a representation for landmark data is given by the Euclidean distance matrix (EDM). In the following, this representation is described in detail and a method is introduced that uses the EDM to compare forms.

Euclidean distance matrix representation

Suppose that the object under study is two dimensional and has K landmarks. Consider

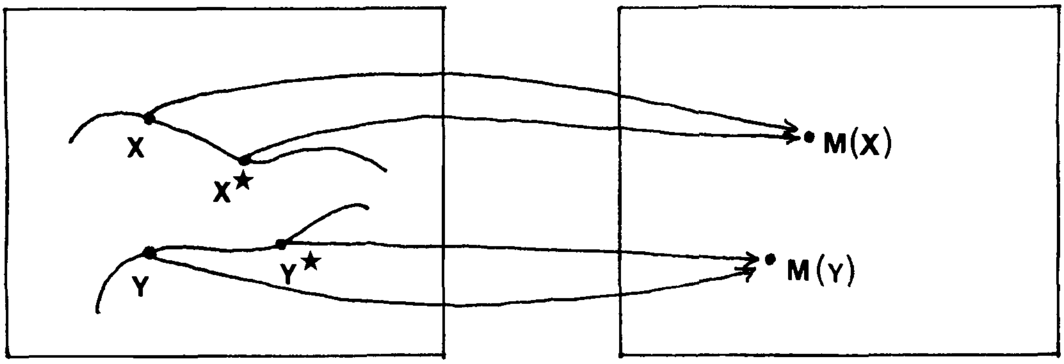

Fig. 2. A pictorial representation of the action of a Maximal Invariant. Maximal invariant $M(\cdot)$ maps all the points on an orbit to a single point in the maximal invariant space.

the following matrix of all possible distances between pairs of landmarks.

$$F = \begin{bmatrix} 0 & d(1,2) & d(1,3) & \cdots & d(1,K) \\ d(2,1) & 0 & \cdots & & d(2,K) \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ d(K,1) & \cdots & & & 0 \end{bmatrix}$$

This is a $K \times K$ symmetric matrix whose $(i,j)^{th}$ element corresponds to the elucidean distance between landmarks i and j on the object. Since this is a matrix of distances, it is clear that it is invariant under translation, rotation, and reflection of the object. The following theorem shows that this representation retains all the information pertaining to the form of an object that is available from landmark data. We call the Euclidean Distance Matrix a Form Matrix.

Theorem 1: Let X be a landmark coordinate matrix corresponding to a given object with K landmarks in $R(2)$. Let $F(X)$ be the form matrix corresponding to the same object. Then given $F(X)$, one can always construct a configuration of K points in $R(2)$, say X', such that X is some translation and rotation/reflection of X'.

Proof: Follows from Theorem 14.1 of Mardia et al. (1979).

This result holds also for three-dimensional objects. In fact, using this result one can characterize the form space of all objects in D-dimensional Euclidean space $R(D)$ with K landmarks as follows:

Theorem 2: The form space of all objects in $R(D)$ with K landmarks is equivalent to the space of all $K \times K$ symmetric positive semidefinite matrices of rank D.

Proof: This again follows from Theorem 14.1 of Mardia et al. (1979).

The above theorem gives a characterization of the form space [or what Kendall (1989) calls a presize and shape space], provided reflection is allowed. Note that this theorem also gives a very nice decomposition of the form space of all figures with K vertices. All K vertex figures on the plane correspond to all $K \times K$ symmetric positive semidefinite matrices of rank 2. All K vertex figures in three dimensions correspond to all $K \times K$ symmetric positive semidefinite matrices of rank 3. Moreover, since a matrix can never be of rank 2 and also of rank 3 these spaces are disjoint. Thus $R[K(K-1)/2]$ space is decomposed into K disjoint subsets—one corresponding to all figures in the plane, one corresponding to all figures in three dimensions, etc. Of course there is one subspace which corresponds to no figures at all. This theorem also suggests that when one wants to choose a statistical model for this set of linear distances, one has to make sure that the sample space has the appropriate rank, either two or three, in order for the samples to correspond to two or three dimensional objects. See Lele and Richtsmeier (1990) for further discussion.

Theorem 3: The form matrix is a maximal invariant under translation, rotation, and reflection.

Proof: Let X and Y be two landmark coordinate matrices corresponding to two objects.

a. It is straightforward to check that

$$F(X) = F(X\,B + 1\,\mathbf{t})$$

for all orthogonal matrices B and (2×2) diagonal matrices $\mathbf{t}$. This follows because distances between landmarks are invariant under these changes. Thus if X and X^* are equivalent, $F(X) = F(X^*)$.

b. To demonstrate maximal invariance, it is necessary to show that if $F(X) = F(Y)$ then $Y = XB + 1\,\mathbf{t}$ for some B and $\mathbf{t}$. This follows from Theorem 1.

Based on this maximal invariant, the Euclidean distance matrix, it can be see that the form of an object with K landmarks can be represented as a point in the L $[= K(K - 1)/2]$-dimensional Euclidean space. In fact it has to belong to (a subset of) the positive quadrant with the axes excluded. Let us call this a "form space."

In order to compare two forms, one naturally needs to define a distance function on the form space. By the very nature of the problem, there are several different choices. Scientific considerations dictate this choice. I suggest the following criteria for such a choice.

Let $D(.,.)$ denote the distance function.

1. Given $F(X)$ and the metric $D(X, Y)$, one should be able to construct $F(Y)$ uniquely, i.e., given figure X and the form difference between X and Y, one should be able to construct figure Y uniquely.

b. The metric $D(.,.)$ should be devoid of any subjective choices of quantities such as loss functions. As shown earlier these choices can be scientifically dangerous.

c. The metric $D(.,.)$ should be interpretable biologically.

Let $F(A)$ and $F(B)$ be two form matrices corresponding to two objects A and B in $R(D)$ with K landmarks. The form difference matrix $D(B,A)$ is defined as follows:

$$D(X, Y) = [F_{ij}(X)/F_{ij}(Y)]$$

where $0/0 = 0$. Note that only the upper diagonal part of this matrix is necessary to study the form difference. This is of size $K(K - 1)/2$.

It is easy to check that the form difference matrix $D(X, Y) = [F_{ij}(X)/F_{ij}(Y)]$ satisfies the above intuitively reasonable requirements. The form difference matrix can also be used for the interpretation and explanation of the underlying biological processes. Each entry in the form difference matrix tells us about the percentage change in the distances between the landmarks involved. How to interpret these changes in terms of the biological processes depends on the problem at hand and the biologists' input becomes important [see Richtsmeier and Lele (1990) for an application].

Given this distance function one can now define equality of forms and equality of shapes in the following manner:

Definition 1: Two objects A and B are said to have the same form if all the off-diagonal entries of $D(B,A)$ are equal to 1.

Definition 2: Two objects A and B are said to have the same shape if all the off-diagonal entries of $D(B,A)$ are equal to c, for some $c > 0$. Or equivalent if $\max_{j>i} D_{ij}/\min_{j>i} D_{ij} = 1$.

Definition 3: If two forms are such that $D(B,A)$ does not satisfy either of the conditions then they have different forms. The ratios smaller than 1 denote shrinking in B as compared to A, and the ratios larger than 1 denote stretching in B as compared to A.

The form matrix or the form difference matrix is fairly large. A natural question is: Can a subset of these landmarks be adequate for comparison of forms? Unfortunately, consideration of only a proper subset of these distances can lead one to erroneous conclusions, as shown in the following example. For the sake of simplicity, suppose we are comparing two objects with three landmarks. Suppose we consider only two distances, say $d(1,2)$ and $d(2,3)$. Based on this subset consisting of two distances only, all the objects in Figure 3 would be considered to have the same form!

Mosimann (1970, 1975a,b) has suggested use of linear distances for studying shapes. However, he neither prescribes (necessarily) the distances between landmarks nor how many distances are needed in order to preserve all the information on the form of the

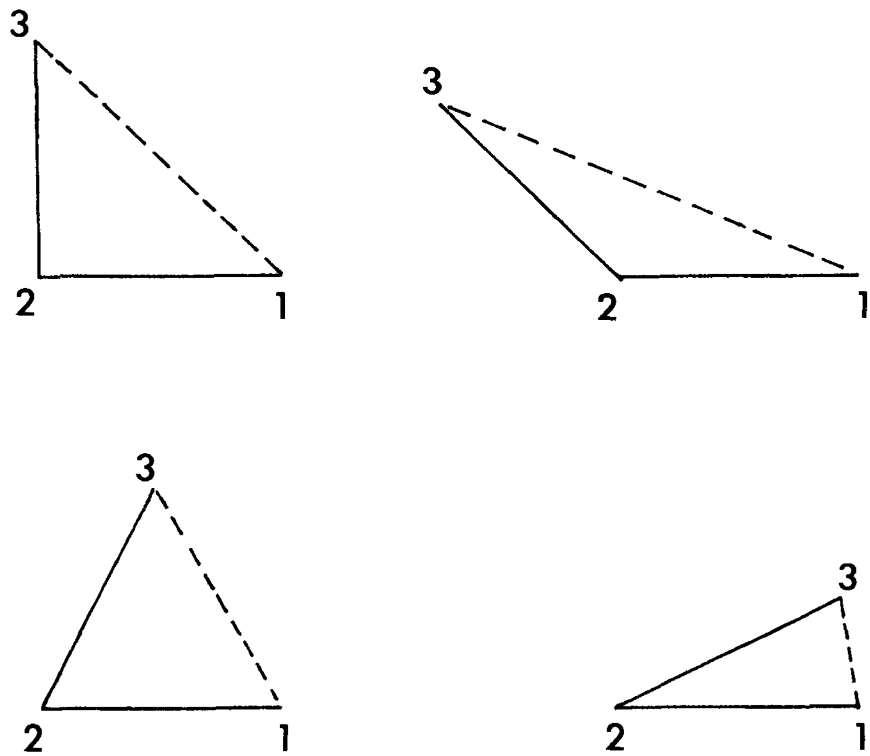

Fig. 3. Inadequacy of the proper subset of all possible distances to represent the form of an object completely: All the triangles in the above figure have the same sides $d(1,2)$ and $d(2,3)$, however they do not have the same form. If one considers only a subset of all possible distances namely $d(1,2)$ and $d(2,3)$, one will declare these triangles to have the same form. The conclusion is that one has to consider all $K(K-1)/2$ distances to completely specify the form of an object with K landmarks.

object as is available in the landmark data. Thus, he ends up with a subset of all possible distances, which could be inadequate as shown above. Strauss and Bookstein (1982) also can be criticized on the same ground.

One cannot claim, however, that all $K(K-1)/2$ distances are necessary to construct the relative locations of K landmarks. For example, it is easy to show that for a two-dimensional object with K landmarks, properly chosen $3(K-2)$ distances are sufficient to construct the relative locations of the landmarks. However, a particular subset may not be sensitive to a given form change. Since one does not know a priori what the form change is, one cannot select a "good" subset of these $K(K-1)/2$ distances. Hence I suggest the use of all the distances.

Shape comparisons

Following the same logic, it is clear that the shape of an object corresponds to the maximal invariant under scaling operation on the form space.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_L)$ be a point in the form space. Let $||x|| = \left(\sum_{i=1}^{L} x_i^2\right)^{1/2}$ denote the norm of this vector and $E(\mathbf{x}) = (\cos^{-1} x_i/||x||, i = 1,2, \ldots, L)$ be the euler angles.

Theorem 4: E(**x**) is a maximal invariant under the group of scaling.

Proof: (i) It is easy to check that $E(\mathbf{x}) = E(c\mathbf{x})$ for all scalar $c > 0$.

(ii) $E(x) = E(y)$ implies that $\mathbf{y} = c\mathbf{x}$ for some scalar $c > 0$.

The second assertion follows because if two points have the same euler angles then they lie on the same ray although they may have different positions. Thus $E(\mathbf{x})$ characterizes the shape of an object.

A better way to represent shape of a configuration of K points would be through all possible angles between triplets of points. However, it is not known how many angles are needed to specify a shape completely and what conditions on the values of these angles would ascertain the existence of a figure in a given Euclidean space. Moreover, in general, angles are more difficult to interpret than are distances. I will not pursue this approach here.

Arbitrariness of the size measures

The above geometry also helps us understand the inherent arbitrariness in the definition of the size measure. One could define shape umambiguously by the euler angles, because of a natural and universally agreed upon mathematical group structure under which shape is invariant, namely that of rotation, reflection, translocation, and scaling. However, there is no such natural and universally agreed upon mathematical group structure under which size is invariant. This leads us to defining a "problem-based natural" group under which size is invariant and hence the existence of a plethora of size measures. For the sake of demonstration, I will consider an unrealistic two-dimensional space and show how the maximal invariants for different size measures look (Fig. 3a–d). The reader can use his/her imagination to draw corresponding pictures in three and higher dimensions.

Note that in Figure 4b–d, all forms that lie on a particular curve have equal size but different shapes, just as all the forms lying on a given ray through the origin have equal shape but different sizes (Fig. 4a). The arbitrariness of the size measure makes the decomposition of form difference into shape difference and size difference arbitrary. The question is: should we formulate our research questions in terms of form rather than size and shape?

In summary, note the following features of EDM analysis for the comparison of forms: (1) The method does not require superimposition and thus there is no need to choose a loss function arbitrarily, (2) the method does not infer anything about how the interior of the object might have defomed. The only real information one has is the relative positions of landmarks, or equivalently the distances between them. It is better if one uses this and only this information to analyze the form difference. Postulating about the relative positions of the interior points which are unobserved is unsound and unnecessary, and (3) the form difference is defined in terms of a maximal invariant and hence it satisfies the Invariance Principle.

However there are two shortcomings of the Euclidean Distance Matrix approach:

1. The form matrix and the form difference matrix are very large, making interpretation difficult. However one can arrange this matrix in an increasing or decreasing order. The landmarks corresponding to the two extremes, small and large ratios, are important biologically. See Richtsmeier and Lele (1990) for an illustration of such analysis.
2. The form difference cannot be represented graphically. However, this seems to be due to the nature of the problem. All the methods that can represent the form difference pictorially seem to resort to some kind of subjective choice such as the nature of the transformation or the loss function.

CONCLUSIONS

The conclusions of this discussion are as follows:

1. The method of superimposition for comparing shapes is subjective. Almost any theory can be supported by choosing convenient loss functions. This is demonstrated through examples. I feel that this subjectivity is scientifically dangerous. Similar criticisms apply to finite element scaling analysis and the use of homology functions.
2. Consideration of the invariance principle leads one to the Euclidean distance matrix representation of the object. The same consideration leads to certain definitions of form difference and shape difference. These are biologically interpretable.
3. While the form difference matrix is large, the necessity of considering the complete matrix and not a subset of it is demonstrated. Methods to extract biologically relevant information from this matrix merit development.
4. Statistical testing based on these invariant quantities merits further study.

Lastly, I would like to mention that no approach is devoid of shortcomings and counterexamples. They do not necessarily make the approach obsolete or nonsensical. However, a researcher should be aware of the merits and demerits of these approaches when drawing conclusions of scientific importance.

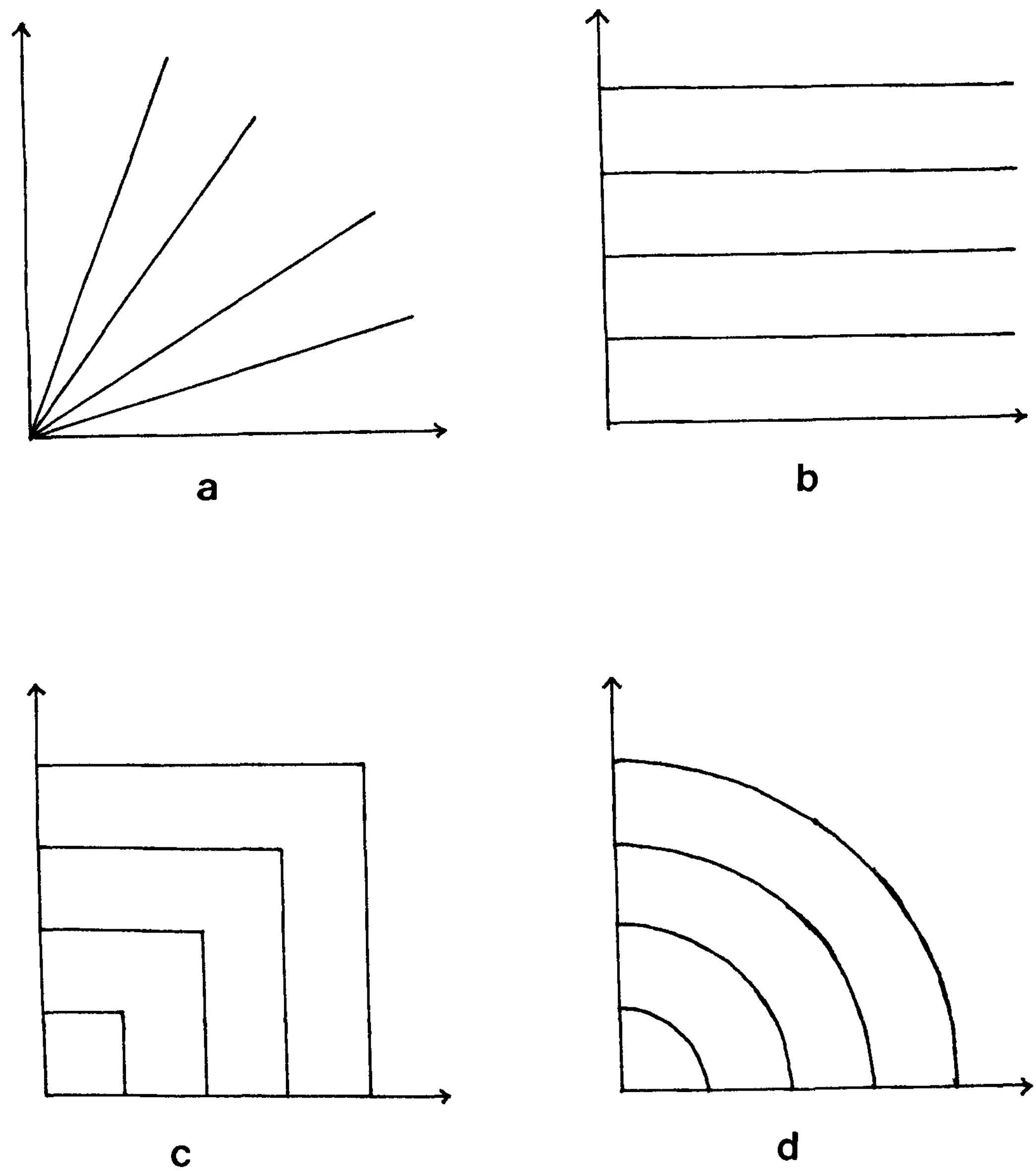

Fig. 4. Maximal invariants under different mathematical groups. (a) The maximal invariants under the group of scaling. All the forms on a given curve have the same shape but different sizes. (b) The maximal invariant when size of the form (x,y) is defined to be y. (c) The maximal invariant when size is $\max(x,y)$; (d) The maximal invariant when size measure is $\sqrt{x^2 + y^2}$. In (b), (c), and (d) the forms on the same curve have equal size but different shapes.

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