Section 4.6 Change of Basis

Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for a finite dimensional vector space $V$. Let $v \in V$ then we can express $v$ as $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$. Recall the coordinate vector of $v$, which was denoted as $(v)_B = (c_1, c_2, \ldots, c_n)$. In this section we will see how to coordinate vectors are changing with the change of basis.

Example: $B = \{(1, 0), (0, 1)\}$ and $B' = \{(1, 1), (2, 1)\}$ are bases for $\mathbb{R}^2$. Consider $u = (2, 3) \in \mathbb{R}^2$. Then we have

$$(u)_B = (2, 3) \quad \text{and} \quad (u)_{B'} = (4, -1)$$

What is the relation between the coordinates of a vector with respect to different bases?

Solution: If we change the basis for a vector space $V$ from an old basis $B=\{u_1, u_2, u_3, \ldots, u_n\}$ to a new basis $B' = \{u'_1, u'_2, \ldots, u'_n\}$, then for each vector $v$ in $V$, the old coordinate vector $[v]_B$ is related to the new coordinate vector $[v]_{B'}$ by the equation

$$[v]_B = P [v]_{B'}$$

where the columns of $P$ are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of $P$ are

$$[u'_1]_B, [u'_2]_B, \ldots, [u'_n]_B$$

$$P_{B' \rightarrow B} = [[u'_1]_B, [u'_2]_B, \ldots, [u'_n]_B], \quad [v]_B = P_{B' \rightarrow B} [v]_{B'}$$

$$P_{B \rightarrow B'} = [[u_1]_{B'}, [u_2]_{B'}, \ldots, [u_n]_{B'}], \quad [v]_{B'} = P_{B \rightarrow B'} [v]_B$$

Example: Consider the bases $B = \{u_1, u_2, u_3\}$ and $B' = \{u'_1, u'_2, u'_3\}$ for $\mathbb{R}^3$, where $u_1 = (-3, 0, -3), u_2 = (-3, 2, -1), u_3 = (1, 6, -1)$ and $u'_1 = (-6, -6, 0), u'_2 = (-2, -6, 4), u'_3 = (-2, -3, 7)$

(a) Find the transition matrix from $B$ to $B'$

(b) Compute the coordinate vector $[w]_B$ where $w = (-5, 8, -5)$ and use part (a) to compute $[w]_{B'}$.

Solution:

(a) We want $P_{B \rightarrow B'} = [[u_1]_{B'}, [u_2]_{B'}, [u_3]_{B'}]$;

$$[u_1]_B : \text{Consider } u_1 = c_1 u'_1 + c_2 u'_2 + c_3 u'_3, \text{ from here we get a linear system of equations:}$$

$$-6c_1 - 2c_2 - 2c_3 = -3$$

$$-6c_1 - 6c_2 - 3c_3 = 0$$

$$4c_2 + 7c_3 = -3$$
Solving this system gives $c_1 = \frac{3}{4}, c_2 = -\frac{3}{4}, c_3 = 0$, hence $[u_1]_B = (\frac{3}{4}, -\frac{3}{4}, 0)$.

$[u_2]_B :$ Consider $u_2 = c_1 u_1' + c_2 u_2' + c_3 u_3'$, from here we get a linear system of equations:

\[-6c_1 - 2c_2 - 2c_3 = -3\]
\[-6c_1 - 6c_2 - 3c_3 = 2\]
\[4c_2 + 7c_3 = -1\]

Solving this system gives $c_1 = \frac{3}{4}, c_2 = -\frac{17}{12}, c_3 = \frac{2}{3}$, hence $[u_2]_B = (\frac{3}{4}, -\frac{17}{12}, \frac{2}{3})$.

$[u_3]_B :$ Consider $u_3 = c_1 u_1' + c_2 u_2' + c_3 u_3'$, from here we get a linear system of equations:

\[-6c_1 - 2c_2 - 2c_3 = 1\]
\[-6c_1 - 6c_2 - 3c_3 = 6\]
\[4c_2 + 7c_3 = -1\]

Solving this system gives $c_1 = \frac{1}{12}, c_2 = -\frac{17}{12}, c_3 = \frac{2}{3}$, hence $[u_3]_B = (\frac{1}{12}, -\frac{17}{12}, \frac{2}{3})$.

Then $P_{B\rightarrow B'} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ \frac{3}{4} & -\frac{17}{12} & \frac{12}{3} \\ 0 & \frac{12}{3} & \frac{12}{3} \end{bmatrix}$

Question: Can we solve the above three systems at the same time?

(b) $[w]_B :$ Consider $w = c_1 u_1 + c_2 u_2 + c_3 u_3$, from here we get the following linear system

\[-3c_1 - 3c_2 + c_3 = -5\]
\[2c_2 + 6c_3 = 8\]
\[-3c_1 - c_2 - c_3 = -5\]

Solving this system gives $c_1 = 1, c_2 = 1, c_3 = 1$, hence $[w]_B = (1, 1, 1)$.

$[w]_{B'} = P_{B\rightarrow B'}[w]_B = \begin{bmatrix} \frac{19}{12} \\ \frac{12}{3} \\ \frac{4}{3} \end{bmatrix}$ hence $[w]_{B'} = (\frac{19}{12}, \frac{-43}{12}, \frac{4}{3})$

Theorem 4.6.1 If $P$ is the transition matrix from a basis $B'$ to a basis $B$ for a finite dimensional vector space $V$, then $P$ is invertible and $P^{-1}$ is the transition matrix from $B$ to $B'$.

Example: We can find $P_{B' \rightarrow B}$ from the previous example, $P_{B' \rightarrow B} = (P_{B \rightarrow B'})^{-1} = \begin{bmatrix} 0 & -\frac{2}{3} & \frac{-3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{2}{2} & \frac{-4}{2} & \frac{-3}{2} \end{bmatrix}$ and one has $[w]_B = P_{B' \rightarrow B}[w]_{B'} = (1, 1, 1)$
Application: Change of basis is used in JPEG compression: Although there are many technical details we can look at the basic idea: A picture can be described by a matrix where each entry is a pixel.

![Image of JPEG compression](http://en.wikipedia.org/wiki/Image_resolution)

Depending on the color of the image there will be a RGB vector representing the pixel. Lets say the image is represented by $512 \times 512$ pixels. Then this image is divided into $8 \times 8$ smaller parts, below is a $8 \times 8$ gray image.

![JPEG example subimage](http://en.wikipedia.org/wiki/File:JPEG_example_subimage.svg)

This image can be represented by a $8 \times 8$ matrix where each pixel is denoted by a number $0 \geq x \leq 255$, 0 stands for black, 255 stands for white.

$$
\begin{bmatrix}
25 & 15 & 65 & 105 & 45 & 67 & 95 & 100 \\
55 & 60 & 70 & 90 & 120 & 87 & 90 & 110 \\
65 & 67 & 78 & 98 & 56 & 50 & 98 & 65 \\
67 & 95 & 37 & 150 & 200 & 190 & 96 & 60 \\
65 & 57 & 87 & 94 & 110 & 200 & 197 & 80 \\
57 & 69 & 84 & 85 & 80 & 90 & 95 & 70 \\
55 & 60 & 70 & 90 & 120 & 87 & 90 & 65 \\
67 & 95 & 37 & 150 & 200 & 190 & 96 & 60 \\
\end{bmatrix}
$$

Note that color of some pixels of the image are very close to each other, hence in the compression process they may be ignored and we may not notice the difference. We can denote the above matrix as a vector in $\mathbb{R}^{64}$ and we have the standard basis, but for the compression process a new basis is chosen for $\mathbb{R}^{64}$, Fourier basis $\{1, \cos(t), \sin(t), \cos(2t), \sin(2t), \cos(3t), \sin(3t), \ldots\}$. The coordinates of the vector representing $8 \times 8$ matrix relative to new basis will be found and smaller coordinates will be ignored, that is will be set to zero.
Section 4.7: Row Space, Column Space, and Null Space

These spaces are associated to matrices.

**Definition:** For \( A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \)

(a) \( r_1 = [a_{11}, a_{12}, \ldots, a_{1n}], r_2 = [a_{21}, a_{22}, \ldots, a_{2n}], \ldots, r_m = [a_{m1}, a_{m2}, \ldots, a_{mn}] \) in \( \mathbb{R}^n \) are called row vectors of \( A \).

(b) \( c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \) in \( \mathbb{R}^m \) are called column vectors of \( A \).

**Definition:** If \( A \) is an \( m \times n \) matrix, then

(a) Row space of \( A \) is the subspace of \( \mathbb{R}^n \) spanned by the row vectors of \( A \).

(b) Column space of \( A \) is the subspace of \( \mathbb{R}^m \) spanned by the column vectors of \( A \).

(c) Nullspace of \( A \) is the solution space of the homogenous system of equations \( Ax = 0 \).

We will see the relation between these spaces and solution set of \( Ax = b \), and we will find basis for each of these spaces for a given \( A \).

First let us see ways to find a basis for these spaces. For this we will look at the effect of elementary row operations on these spaces.
Theorem 4.7.3: Elementary row operations do not change the nullspace of a matrix.

Note that nullspace of a matrix $A$ is formed from the solution set of $Ax = 0$. We know that applying elementary row operation on an augmented matrix does not change the solution set.

Example: Let $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix}$. Find a basis for the nullspace of $A$.

Solution: We need to find $x = (x_1, x_2, x_3, x_4)$ such that $Ax = 0$. For that form the augmented matrix

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$$

Row operations $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 3 & 1 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \end{bmatrix}$

$\Rightarrow x = (\frac{-1}{4}t, -s - \frac{1}{4}t, t, s)$

$\Rightarrow$ Basis for the nullspace of $A = \{(\frac{-1}{4}, \frac{-1}{4}, 1, 0), (0, -1, 0, 1)\}$

Definition: Dimension of the nullspace of $A$ is called nullity of $A$ and is denoted by $\text{nullity}(A)$.

Example: From the previous example $\text{nullity}(A) = 2$.

Note: $\text{nullity}(A) = \text{number of free variables of the system } Ax = 0$

Theorem 4.7.4: Elementary row operations do not change the row space of a matrix.

Let’s see this Theorem on a $2 \times 2$ matrix

$\bullet$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\xrightarrow{R_1 \leftrightarrow R_2} B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\text{Row}(A) = \text{Span}\{(a, b), (c, d)\} = \text{Row}(B)$

$\bullet$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\xrightarrow{2R_1} B = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$

$r_1 = (a, b), r_2 = (c, d)$ are row vectors of $A$, $\tilde{r}_1 = (2a, 2b) = 2r_1, \tilde{r}_2 = (c, d) = r_2$ are row vectors of $B$. Note that $r_1 = \frac{1}{2}\tilde{r}_1$. Then $\text{Span}\{r_1, r_2\} = \text{Span}\{\tilde{r}_1, \tilde{r}_2\}$ by Theorem 4.2.5.

$\bullet$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\xrightarrow{2R_1 + R_2} B = \begin{bmatrix} a & b \\ c + 2a & d + 2b \end{bmatrix}$

$r_1 = (a, b), r_2 = (c, d)$ are row vectors of $A$, $\tilde{r}_1 = (a, b) = r_1, \tilde{r}_2 = (c + 2a, d + 2b) = r_2 + 2r_1$ are row vectors of $B$. Note that $r_2 = \tilde{r}_2 - 2\tilde{r}_1$. Then $\text{Span}\{r_1, r_2\} = \text{Span}\{\tilde{r}_1, \tilde{r}_2\}$ by Theorem 4.2.5.
Example: Let \( A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix} \). Find a basis for the row space of \( A \).

Solution:

\[
\begin{bmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Row \( (B) = \text{Span}\{(1, -1, 0), (0, 1, 0)\} \) and \( \{(1, -1, 0), (0, 1, 0)\} \) is a linearly independent set. Hence \( \{(1, -1, 0), (0, 1, 0)\} \) forms a basis for the row space of \( A \).

Definition: Dimension of the row space of a matrix \( A \) is called the rank of \( A \) and is denoted by \( \text{rank}(A) \).

Example: In the previous example \( \text{rank}(A) = 2 \).

Note: Elementary row operations change the column space of a matrix.

\[
A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_2} B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}
\]

Column space of \( A = \text{Span}\{(1, 2), (3, 6)\} = \text{Span}\{(1, 2)\} \). Column space of \( B = \text{Span}\{(1, 0), (3, 0)\} = \text{Span}\{(1, 0)\} \).

Note that \( (1, 2) \) and \( (1, 0) \) are not scalar multiple of each other. Hence \( \text{Span}\{(1, 2)\} \neq \text{Span}\{(1, 0)\} \).

For finding a basis for column space of a matrix we can still use elementary row operations:

**Theorem 4.7.6:** If \( A \) and \( B \) are row equivalent matrices, then:

(a) A given set of column vectors of \( A \) is linearly independent if and only if the corresponding column vectors of \( B \) are linearly independent.

(b) A given set of column vectors of \( A \) forms a basis for the column space of \( A \) if and only if the corresponding column vectors of \( B \) form a basis for the column space of \( B \).

**Theorem 4.7.5:** If a matrix \( R \) is in row echelon form, then the row vectors with the leading 1’s (the nonzero row vectors) form a basis for the row space of \( R \), and the column vectors with the leading 1’s of the row vectors form a basis for the column space of \( R \).

Example: Find a basis for the row space and column space of the matrix

\[
A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
Solution: Basis for row space =\{(1, 2, 4, 5), (0, 1, -3, 0), (0, 0, 1, -3), (0, 0, 0, 1)\},
Basis for the column space=\{(1, 0, 0, 0), (2, 1, 0, 0, 0), (4, -3, 1, 0, 0), (5, 0, -3, 1, 0)\}

Example: Find a basis for the row space and the column space of the matrix
\[
A = \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & -1 & 3 & -1 \\
2 & 5 & -3 & 3
\end{bmatrix}
\]

Solution:
\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & -1 & 3 & -1 \\
2 & 5 & -3 & 3
\end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & -3 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Basis for row space =\{(1, 2, 0, 1), (0, 1, -3, 1)\}, Basis for column space=\{(1, 0, 2), (2, -1, 5)\}

Observation: Dimension of row space =Dimension of column space.

Section 4.8, Rank, Nullity, and the Fundamental Matrix Spaces

Theorem 4.8.1: If A is any matrix, then the row space and column space of A have the same dimension.

Idea: dimension of column space = number of columns with leading 1’s in the reduced row echelon form
dimension of row space = number of rows with leading 1’s in the reduced row echelon form

Recall rank(A)=dimension of row space of A=dimension of column space of A

Theorem 4.8.7: If A is any matrix, then rank(A)=rank(A^T).

Proof: rank(A)=dim( row space of A) = dim(column space of A^T)=rank(A^T).

Recall nullity(A)=dimension of the solution space Ax=0.

Theorem 4.8.3: If A is an \(m \times n\) matrix, then
(a) \(\text{rank}(A)\) = the number of leading variables in the general solution of \(Ax=0\).
(b) \(\text{nullity}(A)\) = the number of parameters (free variables) in the general solution of \(Ax=0\).

Example: Find the rank and nullity of the matrix
\[
A = \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & -1 & 3 & -1 \\
2 & 5 & -3 & 3
\end{bmatrix}
\]
Solution:

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & -1 & 3 & -1 \\
2 & 5 & -3 & 3
\end{bmatrix}
\]

Row operations \[\Rightarrow\]

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & -1 & 3 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[\Rightarrow\text{rank}=2,\]

For the nullity consider \(Ax=0,\)

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & -1 & 3 & -1 & 0 \\
2 & 5 & -3 & 3 & 0
\end{bmatrix}
\]

Row operations \[\Rightarrow\]

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & -1 & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[\Rightarrow x = s(1, -1, 0, 1) + t(-6, 3, 1, 0) \Rightarrow \text{nullity}=2\]

*Observation*: \(\text{rank}+\text{nullity}=\text{number of columns of } A\)

**Theorem 4.8.2**: (Dimension Theorem for Matrices) If \(A\) is a matrix with \(n\) columns, then

\[\text{rank}(A) + \text{nullity}(A) = n\]

*Idea*: If \(A\) has \(n\) columns then \(Ax=0\), \(x\) has \(n\) components, that is we have \(n\) unknowns

\[\Rightarrow\text{number of leading variables}(=\text{rank}(A)) + \text{number of free variables}(=\text{nullity}(A)) = n\]