Some Exceptional Configurations

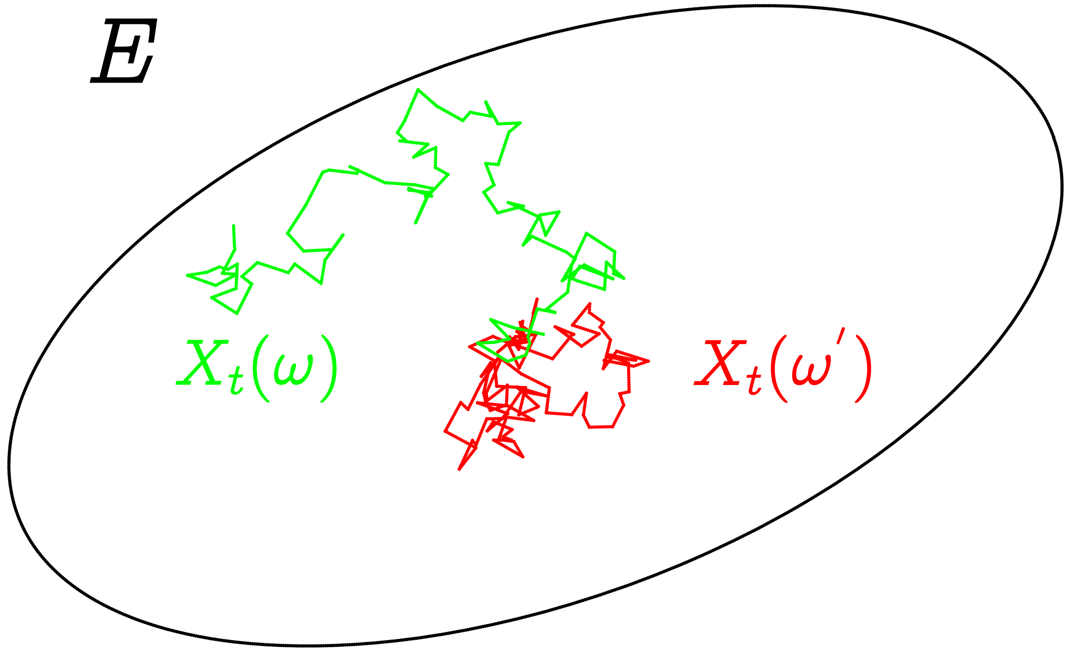
Byron Schmuland
University of Alberta, Canada

H. Osada: "Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions" *CMP 176*, 1996

M.W. Yoshida: "Construction of infinite dimensional interacting diffusion processes through Dirichlet forms" *PTRF 106*, 1996

S. Albeverio, Yu. G. Kondratiev, and M. Röckner: "Analysis and geometry on configuration spaces" *JFA 154*, 1998

"Analysis and geometry on configuration spaces: The Gibbsian case" *JFA 157*, 1998



(X_t, P_γ) : diffusion process on E with invariant measure μ

$(\mathcal{E}, D(\mathcal{E}))$: Dirichlet form on $L^2(E;\mu)$

For $u : E \to \mathbb{R}$ we have

$$\mathcal{E}(u,u) = \lim_{t \downarrow 0} \frac{1}{2t} E_\mu \left[(u(X_t) - u(X_0))^2 \right]$$

$u : E \longrightarrow \mathbb{R}$ is *quasi continuous* if

$$P_\gamma(t \mapsto u(X_t) \text{ continuous }) = 1$$

$N \subseteq E$ is *exceptional* if

$$P_\gamma(X_t \notin N \text{ for all } t) = 1$$

That is, $\mu(N) = 0$ and 1_N is quasi continuous

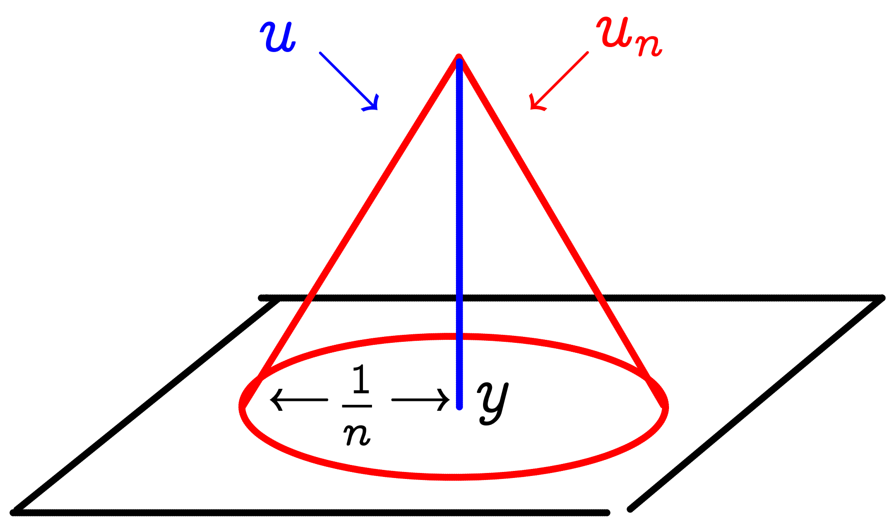
Lemma QC: If $u_n \in D(\mathcal{E})$ are quasi continuous, $u_n \longrightarrow u$ pointwise, and

$$\sup_n \mathcal{E}(u_n, u_n) < \infty,$$

then u is quasi continuous

$$E = \mathbb{R}^d, \quad \mu = \text{Lebesgue measure}, \quad X_t = \text{Brownian motion}$$

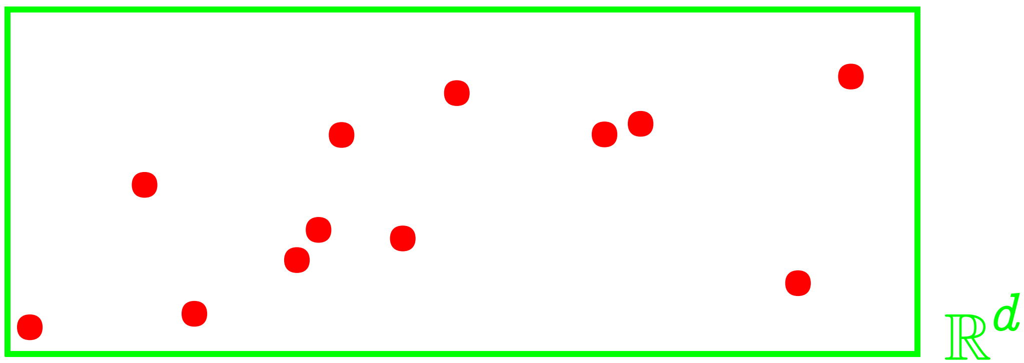
$$\mathcal{E}(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v \rangle \, d\mu, \quad N = \{y\}$$



$$\mathcal{E}(u_n,u_n) = \frac{1}{2}\int_{B(y,1/n)} \|\nabla u_n\|^2 \, d\mu = cn^{-d}n^2$$

Configuration space Γ

$$\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty, \text{ compact } K\}$$



Each γ is a Radon measure, Γ has topology of vague convergence. For $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\gamma \in \Gamma$ define

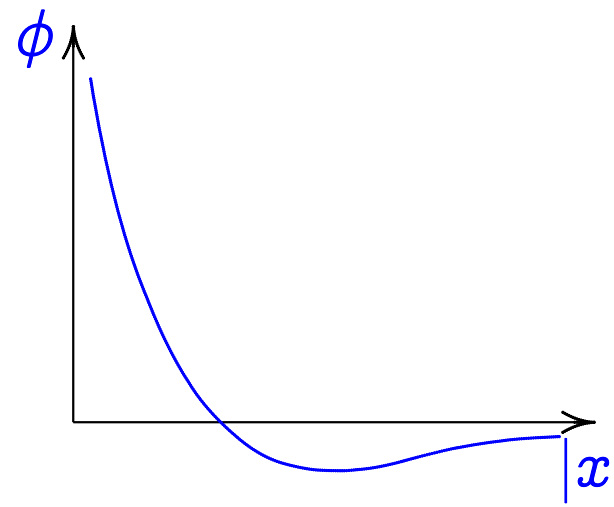
$$\langle \varphi, \gamma \rangle = \int \varphi(x)\, \gamma(dx) = \sum_{x \in \gamma} \varphi(x)$$

Gibbs measures on Γ

Poisson measure π_z characterized by

$$\int_\Gamma \exp(\langle \varphi, \gamma \rangle)\, \pi_z(d\gamma) = \exp\left[z \int (e^\varphi - 1) d\lambda \right]$$

where λ is Lebesgue measure

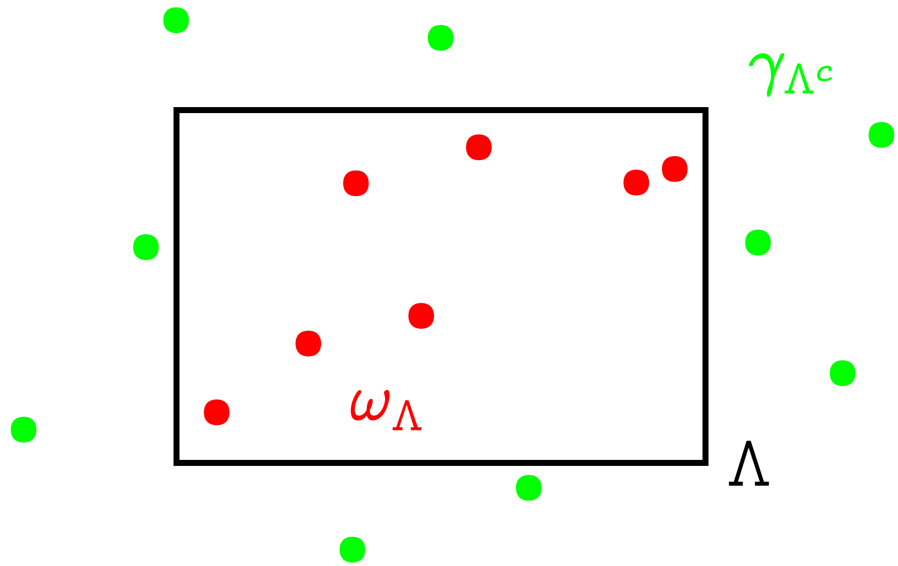


- Poisson measure π_z
- pair potential $\phi : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$
- energy $E_\Lambda^\phi(\gamma) = \sum \phi(x - y)$

  sum over $\{x, y\} \subset \gamma, \{x, y\} \cap \Lambda \neq \emptyset$

For a Gibbs measure μ the conditional distribution of the part of the configuration inside Λ, given γ outside Λ:

$$\sim \exp(-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda))\, \pi_z(d\omega)$$



Test functions on Γ

$$\mathcal{F}C_b^\infty := \{\, u : u(\gamma) = g(\langle \varphi_1, \gamma\rangle, \langle \varphi_2, \gamma\rangle, \ldots, \langle \varphi_n, \gamma\rangle) \\ \varphi_i \in C_0^\infty(\mathbb{R}^d) \text{ and } g \in C_b^\infty(\mathbb{R}^n) \,\}$$

$$\left(\nabla^\Gamma u\right)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(\langle \varphi_1, \gamma\rangle, \langle \varphi_2, \gamma\rangle, \ldots, \langle \varphi_n, \gamma\rangle\right) \nabla \varphi_i(x)$$

Define the *square field* by

$$S(u,v)(\gamma) := (\nabla^\Gamma u(\gamma;\cdot), \nabla^\Gamma v(\gamma;\cdot))_{L^2(\mathbb{R}^d \to \mathbb{R}^d;\gamma)}$$

Dirichlet form on Γ

pre-Dirichlet form on $L^2(\Gamma;\mu)$

$$\mathcal{E}(u,v)=\frac{1}{2}\int_\Gamma S(u,v)(\gamma)\,\mu(d\gamma)$$

for $u,v\in\mathcal{F}C_b^\infty$

Closable? Yes!

The closure $(\mathcal{E},D(\mathcal{E}))$ is a Dirichlet form

Markov process on Γ

Is $(\mathcal{E}, D(\mathcal{E}))$ quasi-regular?

Two potential problems:

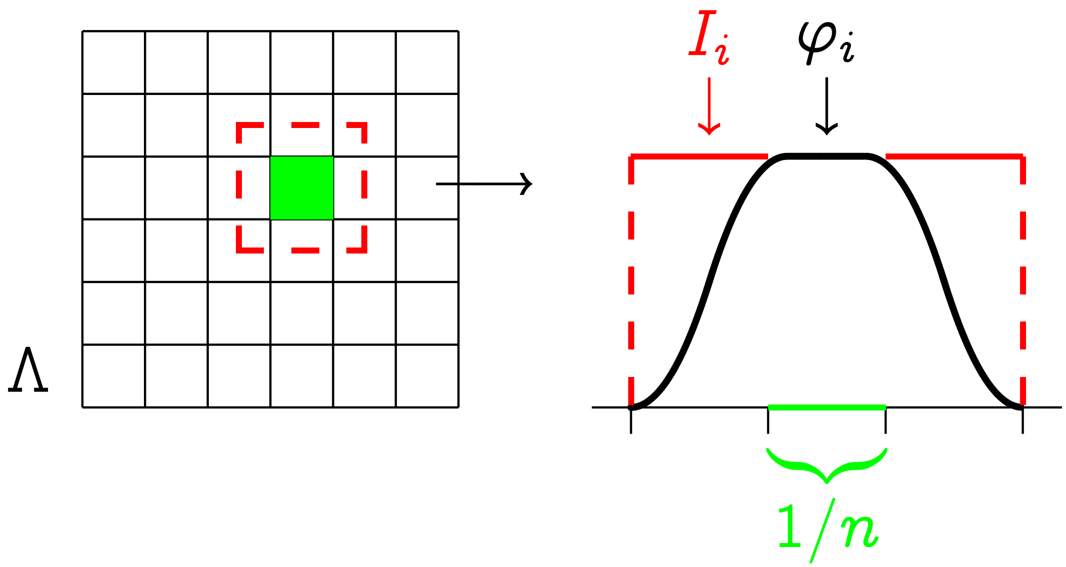
- implosion
- collision

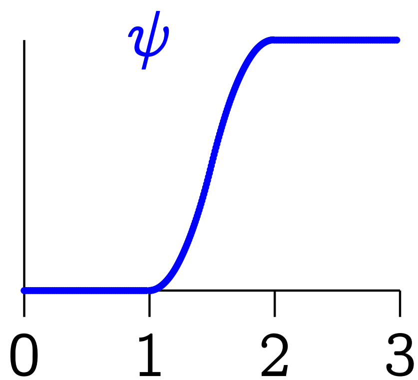
$(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular on

$$\ddot{\Gamma}_{\mathbb{R}^d} := \{\mathbb{Z}_+ \cup \{\infty\}\text{-valued Radon measures on } \mathbb{R}^d\}$$

$(X_t)_{t\geq 0}$ is a $\ddot{\Gamma}_{\mathbb{R}^d}$-valued process

Proposition 1: $\ddot{\Gamma}_{\mathbb{R}^d} \setminus \Gamma_{\mathbb{R}^d}$ is exceptional if $d \geq 2$





$$u_n(\gamma) = \psi\left(\sup_i \langle \varphi_i, \gamma \rangle\right)$$

$$S(\langle\varphi,\cdot\rangle,\langle\varphi,\cdot\rangle)(\gamma)=\langle\|\nabla\varphi\|^2,\gamma\rangle$$

$$S(u\vee v,u\vee v)(\gamma)\leq S(u,u)\vee S(v,v)$$

$$\begin{aligned}
S(u_n,u_n)(\gamma) &\leq c1_{[\sup_i\langle I_i,\gamma\rangle\geq 2]}\sup_i\langle\|\nabla\varphi_i\|^2,\gamma\rangle\\
&\leq c1_{[\sup_i\langle I_i,\gamma\rangle\geq 2]}\,n^2\sup_i\langle I_i,\gamma\rangle\\
&\leq cn^2\sum_i 1_{[\langle I_i,\gamma\rangle\geq 2]}\langle I_i,\gamma\rangle
\end{aligned}$$

Free case: (Poisson measure μ)

$$\langle I_i, \gamma \rangle \sim \text{ Poisson}(z\lambda(I_i))$$

$$\int_{[\langle I_i,\gamma\rangle \geq 2]} \langle I_i, \gamma \rangle \, \mu(d\gamma) \leq (z\lambda(I_i))^2$$

$$\mathcal{E}(u_n, u_n) \leq cn^2 \sum_i (n^{-d})^2 \sim cn^{2-d}$$

Gibbs case: (Ruelle measure μ)

There is ξ so that for bounded Λ

$$\left(\frac{d\text{Gibbs}}{d\text{Poisson}_\xi} \right) \Bigg|_{\mathcal{F}_\Lambda} \leq c(\xi, \Lambda)$$

so Proposition 1 holds.

LLN and LIL on Γ

Joint work with Wei Sun

$$G_n = [-n, n]^d, \mu_n = E_\mu(\gamma(G_n)), \sigma_n^2 = \mathrm{Var}_\mu(\gamma(G_n))$$

$$LLN := \left\{ \gamma \in \ddot{\Gamma}_{\mathbb{R}^d} : \lim_{n\to\infty} \frac{\gamma(G_n)}{\mu_n} = 1 \right\}$$

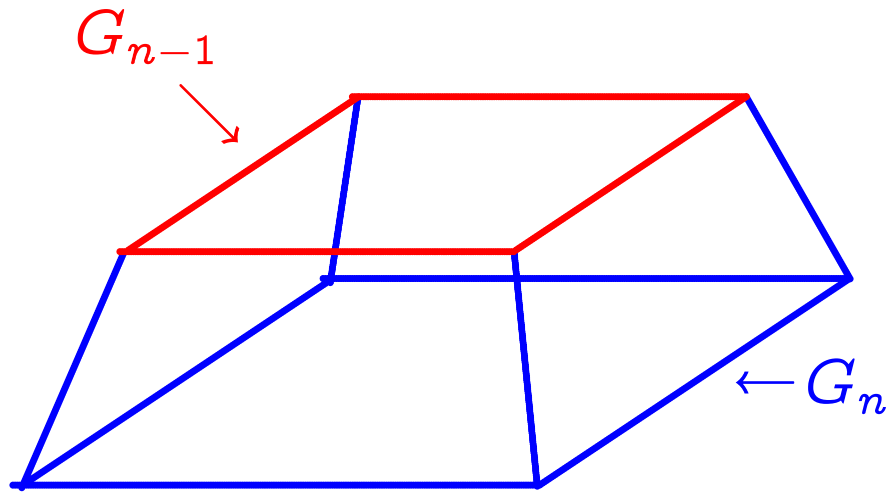
$$LIL := \left\{ \gamma \in \ddot{\Gamma}_{\mathbb{R}^d} : \limsup_{n\to\infty} \frac{\gamma(G_n) - \mu_n}{(2\sigma_n^2 \log\log \sigma_n^2)^{1/2}} = 1 \right\}$$

In the free case, with Poisson measure μ, it is easy to show that $\mu(\mathrm{LLN}^c) = 0$ and $\mu(\mathrm{LIL}^c) = 0$.

In the Gibbs case, with Ruelle measure μ, for small values of the activity parameter z, μ is translation invariant on $\mathbb{R}^d$ and we have showed that $\mu(\mathrm{LLN}^c) = 0$ and $\mu(\mathrm{LIL}^c) = 0$.

Proposition 2: Under the conditions above, the sets LLN^c and LIL^c are exceptional.

$$\varphi_n$$



$$u_n(\gamma) := \frac{\langle \varphi_n, \gamma \rangle}{\lambda(G_n)}$$

Bounding the square field gives

$$S(u_n, u_n)(\gamma) \leq c \, \frac{\gamma(G_n)}{\lambda(G_n)^2}$$

Write G_n as the union of cubes Q_m of size one, so $1_{G_n} = \sum_{m=1}^{\lambda(G_n)} 1_{Q_m}$. Then

$$\frac{1_{G_n}}{\lambda(G_n)^2} \leq \sum_{m=1}^{\lambda(G_n)} \frac{1_{Q_m}}{m^2} \leq \sum_{m=1}^{\infty} \frac{1_{Q_m}}{m^2},$$

so

$$\begin{aligned}
\int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \frac{\gamma(G_n)}{\lambda(G_n)^2}\,\mu(d\gamma) &= \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \frac{1_{G_n}(x)}{\lambda(G_n)^2}\,\gamma(dx)\,\mu(d\gamma) \\
&\leq \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \int_{\mathbb{R}^d} \sum_{m=1}^{\infty} \frac{1_{Q_m}(x)}{m^2}\,\gamma(dx)\,\mu(d\gamma) \\
&= \sum_{m=1}^{\infty} \frac{E_\mu(\gamma(Q_1))}{m^2} < \infty
\end{aligned}$$

By Lemma QC, the limit $u := \limsup_n u_n$ belongs to $D(\mathcal{E})$ and is quasi-continuous.

$$\begin{aligned} u(\gamma) &= \limsup_{n\to\infty} \gamma(G_n)/\lambda(G_n) \\ v(\gamma) &:= \liminf_{n\to\infty} \gamma(G_n)/\lambda(G_n) \\ w(\gamma) &:= E_\mu(\gamma(Q_1)) \end{aligned}$$

are quasi-continuous. Since u, v, and w agree μ-almost everywhere, they must agree except on an exceptional set.

LLN^c is exceptional!