A Dirichlet form primer.

Byron Schmuland
Department of Statistics and Applied Probability
University of Alberta
Edmonton, Alberta, Canada
T6G 2G1

A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, like a Feller semigroup, is an analytic object that can be used to construct and study a certain Markov process $\{X_t\}_{t\geq 0}$. Unlike the Feller semigroup approach, which uses a pointwise analysis, the Dirichlet form approach uses a quasi-sure analysis, meaning that we are permitted to ignore certain exceptional sets which are not visited by the process. This slight ambiguity, where some of the definitions and results only hold quasi-everywhere, has certain advantages and certain disadvantages. On the one hand, even when studying as familiar a process as Brownian motion in $\mathbb{R}^n$, we can do some things more generally. For instance, M. Fukushima [F 80; Example 5.1.1.] shows that when $n \geq 2$ and for any $\alpha \in \mathbb{R}$, the additive functional $A_t(\omega) = \int_0^t |X_s(\omega)|^\alpha \, ds$ makes perfect sense in the context of Dirichlet forms, even though when $\alpha \leq -2$ we have

$$P_0\Big(\int_0^t |X_s(\omega)|^\alpha \, ds = \infty \Big) = 1. \tag{1}$$

This is because the state $\{0\}$, which is causing all the headaches, is polar and so can be completely removed from consideration. On the other hand, this slight ambiguity also means that when we use Dirichlet forms to solve stochastic differential equations for example, as in [AR 91], the solutions are defined only from quasi-every starting point in the state space, rather than from every starting point. The absence of pointwise results is the price we pay for the increased generality of cases that can be covered using Dirichlet form theory.

Dirichlet forms have emerged as a powerful tool of stochastic analysis especially in the case where the state space is infinite dimensional, where it is useful to throw away (topologically) large exceptional sets, even for very standard examples. In this paper we shall look at some recent developments in the general theory of Dirichlet forms, especially at those aspects that allow us to consider infinite dimensional cases. In particular, we will try to explain the condition of *quasi-regularity*, how it is defined and what it implies for a Dirichlet form. Our main reference is the recent book by Z.M. Ma and M. Röckner [MR 92] whose treatment and notation we will try to follow throughout. The reader interested in other recent work in the area is encouraged to look, as well, at [AM 91 a,b], [AMR 92,93], [AR 89,90,91], [BH 91], [RZ 92], and the references therein.

Let E be a Hausdorff topological space equipped with its Borel σ-field $\mathcal{B}(E)$, and m a σ-finite Borel measure on E.

Definition 1. A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *closed form* on (real) $L^2(E; m)$ if $D(\mathcal{E})$ is a dense linear subspace of $L^2(E; m)$ and if $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a bilinear form such that the following conditions hold:

(i) $\mathcal{E}(u,u) \geq 0$ for all $u \in D(\mathcal{E})$.

(ii) $D(\mathcal{E})$ is a Hilbert space when equipped with the inner product $\tilde{\mathcal{E}}_1(u,v) := (1/2)\{\mathcal{E}(u,v) + \mathcal{E}(v,u)\} + (u,v)_{L^2}$.

Definition 2. Given a closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ we define we following closed forms all of whose domains coincide with $D(\mathcal{E})$:

(i) $\tilde{\mathcal{E}}(u,v) = (1/2)\{\mathcal{E}(u,v) + \mathcal{E}(v,u)\}$.

(ii) $\widehat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$.

(iii) $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2}$.

Definition 3. A closed form $(\mathcal{E}, D(\mathcal{E}))$ is called *coercive* if it satisfies the sector condition (see 9 below).

Definition 4. A coercive, closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called a *Dirichlet form* if it has the following (unit) contraction property: for all $u \in D(\mathcal{E})$, we have $u^+ \wedge 1 \in D(\mathcal{E})$ and

$$\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0 \quad \text{and} \quad \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0. \tag{2}$$

Our hope would be to try to establish a relation of the following type between this form $\mathcal{E}$ and some Markov process $\{X_t\}$.

Definition 5. A right process $\mathbf{M}$ (cf. [Sh 88]) with state space E and transition semigroup $(p_t)_{t>0}$ is said to be *associated with* $(\mathcal{E}, D(\mathcal{E}))$ if $p_t f$ is an m-version of $T_t f$ for all $t > 0$ and every bounded, Borel measurable f in $L^2(E;m)$.

The construction of the process associated with a given Dirichlet form, when it is possible, follows a path that looks something like this:

$$(\mathcal{E}, D(\mathcal{E})) \to (T_t)_{t\geq 0} \to (p_t)_{t\geq 0} \to \{X_t\}_{t\geq 0}. \tag{3}$$

The first step, to get the L^2 generator $(T_t)_{t\geq 0}$ is usually not difficult, and follows from the theory of semigroups of linear operators applied to the the Hilbert space $L^2(E;m)$. Condition (2) guarantees that T_t behaves like a sub-Markov transition kernel in the sense that $0 \leq f \leq 1$ implies $0 \leq T_t f \leq 1$ m-a.e. Also, once you are able to get a reasonable kernel $(p_t)_{t\geq 0}$, the last step follows from standard constructions of a Markov process from $(p_t)_{t\geq 0}$ (or from the resolvent kernel $(R_\alpha)_{\alpha>0}$) on general state spaces, as done for instance in Sharpe's book [Sh 88]. The middle step relates the L^2 semigroup $(T_t)_{t\geq 0}$ with the transition semigroup $(p_t)_{t\geq 0}$ on E, and this is the most difficult, especially if you have a particular process $\{X_t\}$ in mind. It is in this step where your decision on an appropriate state space E and measure m plays the biggest role.

Once such a relationship between $(\mathcal{E}, D(\mathcal{E}))$ and $\mathbf{M}$ has been established we can use analysis of $(\mathcal{E}, D(\mathcal{E}))$ to get information about the behaviour of the process $\{X_t\}$. For instance, if A is a Borel subset of E with $m(A) = 0$, then $p_t 1_A(z) = 0$ m-a.e. $z \in E$. That means from m-almost every starting point, at a fixed time t, we won't catch X_t visiting the set A, i.e., $P_m(X_t \in A) = 0$. A stronger notion of smallness of sets is needed if we want to conclude that X_t absolutely never visits A, i.e., $P_m(\exists t \geq 0$ so that $X_t \in A) = 0$. We will discuss this in Proposition 15.

Until recently, the general theory of Dirichlet forms had been restricted to the case where the underlying space E is locally compact. In M. Fukushima's book [F 80], which is the standard reference in the area, the local compactness is used throughout and is crucial in the construction of the associated Markov processes. Fukushima assumes that E is a locally compact, separable metric space and that m is a positive Radon measure on $\mathcal{B}(E)$ with full support. He then constructs a Markov process, indeed a Hunt process, associated with any symmetric, regular Dirichlet form.

Definition 6. A Dirichlet form is called *symmetric* if $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ for all $u, v \in D(\mathcal{E})$.

Definition 7. A Dirichlet form is called *regular* if $D(\mathcal{E}) \cap C_0(E)$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$, and is uniformly dense in $C_0(E)$. Here $C_0(E)$ is the space of continuous real-valued functions with compact support.

Now the local compactness assumption, of course, eliminates the possibility of using Fukushima's theory in the study of infinite-dimensional processes. Also the symmetry assumption confines us to treating only symmetrizable processes. Nevertheless, in the years since the publication of [F 80] several authors have been able to modify Fukushima's construction in special cases and obtain processes in infinite-dimensional state spaces. (cf. [AH-K 75, 77 a,b], [Ku 82], [AR 89,91], [S 90b], [BH 91], and see also the reference list in [MR 92]). The theory of non-symmetric forms and the construction of the associated process has also been considered by a number of authors (cf. [CaMe 75], [FdL 78], [Le 82], [O 88]). It turns out that while neither assumption can be dropped, both are too strong.

Example 8. Why we can't just drop symmetry.

Let's look at the least symmetric process possible. Consider the space $E = \mathbb{R}$ equipped with the usual topology and with m equal to Lebesgue measure. Let $\{X_t\}$ be the process that justs moves to the right uniformly. That is, $p_t(z, dy) = \epsilon_{z+t}(dy)$. This yields the strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^2(E; m)$ which acts on f by the formula $(T_t f)(z) = f(z+t)$. The generator L is then $Lf = f'$ and $D(L) = \{f \in L^2 :$ f is absolutely continuous and $f' \in L^2\}$. Therefore the usual pre-Dirichlet form defined by $D(\mathcal{E}) = D(L)$ and $\mathcal{E}(f,g) = \int(-Lf)\, g\, dz$, is not closable. In fact, $\mathcal{E}$ is anti-symmetric so the Hilbert space norm in Definition 1 (ii) is just the L^2-norm. An unclosable form fails to give us the kind of probabilistic potential theory needed to connect the form $\mathcal{E}$ and the process $\{X_t\}$. So, in this example, a reasonable Markov process leads to an unreasonable form. This kind of situation can be avoided by assuming the sector condition, which allows us to control the form $\mathcal{E}$ by its symmetric part $\tilde{\mathcal{E}}$.

Condition 9. The substitute for symmetry: The sector condition.

The form $(\mathcal{E}, D(\mathcal{E}))$ satisfies the *sector condition* if there exists a constant $K > 0$ such that

$$|\mathcal{E}_1(u,v)| \leq K\ \mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}, \tag{4}$$

for all $u, v \in D(\mathcal{E})$.

Example 10. Why we can't just drop regularity.

Take $E = [0,1)$ equipped with $m=$ Lebesgue measure, and let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form associated with reflecting Brownian motion on $[0,1]$. That is,

$$\begin{aligned} D(\mathcal{E}) &= \{u : u \text{ is absolutely continuous on } (0,1) \text{ and } u' \in L^2((0,1); dz)\} \\ \mathcal{E}(u,v) &= (1/2) \int u'(z)\, v'(z)\, dz. \end{aligned} \tag{5}$$

This form is not regular, in fact, the closure of $C_0([0,1))$ in $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{1/2})$ consists only of those functions f in $D(\mathcal{E})$ with $f(z) \to 0$ as $z \to 1$. And of course it is impossible to construct a reasonable process on $[0,1)$ associated with the form $(\mathcal{E}, D(\mathcal{E}))$ because the appropriate process is reflecting Brownian motion on $[0,1]$. The space $E = [0,1)$ is adequate to define the form but not as a state space for the process. So this time we had a reasonable Dirichlet form lead to an unreasonable process.

Now in this example the problem is obviously that the point $\{1\}$ is missing from the space, and if we put it back, we recover the usual regular form on $E = [0,1]$ corresponding to reflecting Brownian motion. However, the problem of "missing boundary points" can occur even when the space E is topologically complete. This can happen when the measure m lives on one Hilbert space, but the process $\{X_t\}$ needs an even bigger Hilbert space as a state space. Such an example is discussed in [RS 93]. We'd like to find a condition that is not as restrictive as regularity, but which would let us know that our space is large enough to accommodate the associated process. But before we can get to our substitute for regularity we need to learn a bit more about exceptional sets for Dirichlet forms.

Definition 11.

(i) For a closed subset $F \subseteq E$ we define

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) |\;\; u = 0 \;\; m\text{−}a.e. \;\text{ on } E \backslash F\}. \tag{6}$$

Note that $D(\mathcal{E})_F$ is a closed subspace of $D(\mathcal{E})$.

(ii) An increasing sequence $(F_k)_{k \in \mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$-*nest* if $\cup_{k \geq 1} D(\mathcal{E})_{F_k}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$.

(iii) A subset $N \subset E$ is called $\mathcal{E}$-*exceptional* if $N \subseteq \cap_k F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$. A property of points in E holds $\mathcal{E}$-*quasi-everywhere* (abbreviated $\mathcal{E}$-q.e.), if the property holds outside some $\mathcal{E}$-exceptional set. It can be seen that every $\mathcal{E}$-exceptional set has m-measure zero.

(iv) An $\mathcal{E}$-q.e. defined function $f : E \to \mathbb{R}$ is called $\mathcal{E}$-*quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$ so that $f|_{F_k}$ is continuous for each $k \in \mathbb{N}$.

(v) Let $(f_n)_{n \in \mathbb{N}}$ and f be ($\mathcal{E}$-q.e. defined) functions on E. We say that $(f_n)_{n \in \mathbb{N}}$ converges $\mathcal{E}$-*quasi-uniformly* to f if there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$ such that $f_n \to f$ uniformly on each F_k.

The following result is very important in Dirichlet form analysis (cf. [MR 92; Chapter III, Proposition 3.5.]).

Lemma 12. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(E;m)$ and suppose that $(u_n)_{n\in\mathbb{N}} \in D(\mathcal{E})$, having $\mathcal{E}$-quasi-continuous m-versions $(\tilde{u}_n)_{n\in\mathbb{N}}$, converges to $u \in D(\mathcal{E})$ in $\tilde{\mathcal{E}}_1^{1/2}$-norm. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ of u so that $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ converges $\mathcal{E}$-quasi-uniformly to $\tilde{u}$.*

Condition 13. The substitute for regularity: Quasi-regularity.

A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called *quasi-regular* if:

(QR1) There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(QR2) There exists an $\tilde{\mathcal{E}}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(QR3) There exists a countable collection $(u_n)_{n\in\mathbb{N}} \in D(\mathcal{E})$, having $\mathcal{E}$-quasi-continuous m-versions $(\tilde{u}_n)_{n\in\mathbb{N}}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $(\tilde{u}_n)_{n\in\mathbb{N}}$ separates the points of $E \setminus N$.

The fundamental existence result in the framework of quasi-regular forms is found in [MR 92; Chapter IV, Theorem 6.7.] and it says the following:

Theorem 14. *Let E be a metrizable Lusin space. Then a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is quasi-regular if and only if there exists a pair $(\mathbf{M}, \widehat{\mathbf{M}})$ of normal, right continuous, strong Markov processes so that $\mathbf{M}$ is associated with $\mathcal{E}$, and $\widehat{\mathbf{M}}$ is associated with $\widehat{\mathcal{E}}$.*

This says that the class of quasi-regular Dirichlet forms is the correct setting for the study of those forms associated with nice Markov processes. L. Overbeck and M. Röckner [OvR 93] have recently proved a one-sided version of the existence result for quasi-regular *semi*-Dirichlet forms; i.e., for those closed, coercive forms $(\mathcal{E}, D(\mathcal{E}))$ such that only the first relation in (2) holds. In this case, of course, we don't get a pair of processes but only the process $\mathbf{M}$.

The next result is the heart of the matter, as it establishes the connection between the probabilistic potential theory of the process $\mathbf{M}$ and the analytic potential theory of the form $(\mathcal{E}, D(\mathcal{E}))$. (cf. [MR 92; Chapter IV, Theorem 5.29.]).

Proposition 15. *Suppose the right process $\mathbf{M}$ is associated with the quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Then a subset N of E is $\mathcal{E}$-exceptional if and only if $N \subset \tilde{N}$ where $\tilde{N}$ is a Borel set such that*

$$P_m(\tau_{\tilde{N}} < \zeta) = 0, \tag{7}$$

where ζ is the lifetime of the process and τ means the touching time of the set, i.e., for any Borel set B we define $\tau_B(\omega) := \inf\{0 \le t < \zeta(\omega) \; : \; X_t(\omega) \in B \text{ or } X_{t-}(\omega) \in B\} \wedge \zeta(\omega)$.

It is easy to see that this new concept of a quasi-regular Dirichlet form includes the classical concept of a regular Dirichlet form. This follows from the next proposition (cf. [MR 92; Chapter IV, Section 4a]).

Proposition 16. *Assume E is a locally compact, separable, metric space and m is a positive Radon measure on $\mathcal{B}(E)$. If $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(E;m)$, then $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.*

Proof. We only show (QR1), as (QR2) and (QR3) are easy exercises. By the topological assumptions on E, we may write $E = \cup_{k=1}^{\infty} F_k$, where $(F_k)_{k\in\mathbb{N}}$ is an increasing sequence of compact sets in E so that F_k is contained in the interior of F_{k+1} for all $k \geq 1$. It is then easy to see that

$$C_0(E) \cap D(\mathcal{E}) \subseteq \cup_{k=1}^{\infty} D(\mathcal{E})_{F_k}, \tag{8}$$

which concludes the proof.

By Theorem 14, we know that if we have a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, even on an infinite dimensional Banach space E, and we want to show the existence of an associated process, all we have to do is prove that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Typically the form $(\mathcal{E}, D(\mathcal{E}))$ is defined as the closure of some pre-Dirichlet form over a core of continuous functions that separates points in E. So usually the conditions (QR2) and (QR3) of Condition 13 are easily checked. It is (QR1) that is usually most troublesome, and that is because it is not always easy to come up an explicit sequence of compact sets that do the trick, especially in infinite dimensions. We note that the condition (QR1) is equivalent to the tightness of a certain capacity defined on subsets of E (cf. [LR 91], [RS 92], [S 92]). The following result is based on Proposition 3.1 of [RS 92], and it shows when a Dirichlet form of gradient type defined on a Banach space is quasi-regular (see also [MR 92; Chapter IV, Section 4b]).

Example 17. Gradient-type forms on an infinite dimensional Banach space.

Let E be a (real) separable Banach space, and m a finite measure on $\mathcal{B}(E)$ which charges every weakly open set. Define a linear space of functions on E by

$$\mathcal{F}C_b^{\infty} = \{f(l_1, \ldots, l_m) : m \in \mathbb{N}, f \in C_b^{\infty}(\mathbb{R}^m), l_1, \ldots, l_m \in E'\}. \tag{9}$$

Here $C_b^{\infty}(\mathbb{R}^m)$ denotes the space of all infinitely differentiable functions on $\mathbb{R}^m$ with all partial derivatives bounded. By the Hahn-Banach theorem, $\mathcal{F}C_b^{\infty}$ separates the points of E. The support condition on m means that we can regard $\mathcal{F}C_b^{\infty}$ as a subspace of $L^2(E;m)$, and by a monotone class argument you can show that it is dense in $L^2(E;m)$. Define for $u \in \mathcal{F}C_b^{\infty}$ and $k \in E$,

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds}\, u(z + sk)|_{s=0}, \ z \in E. \tag{10}$$

Observe that if $u = f(l_1, \ldots, l_m)$, then

$$\frac{\partial u}{\partial k} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(l_1, \ldots, l_m)\, {}_{E'}\langle l_i, k\rangle_E, \tag{11}$$

which shows us that $\partial u/\partial k$ is again a member of $\mathcal{F}C_b^\infty$. Also let us assume that there is a separable real Hilbert space $(H, \langle\, , \rangle_H)$ densely and continuously embedded into E. Identifying H with its dual H' we have that

$$E' \subset H \subset E \quad \text{densely and continuously,} \tag{12}$$

and ${}_{E'}\langle\, , \rangle_E$ restricted to $E' \times H$ coincides with $\langle\, , \rangle_H$. Observe that by (11) and (12), for $u \in \mathcal{F}C_b^\infty$ and fixed $z \in E$, the map $k \to (\partial u/\partial k)(z)$ is a continuous linear functional on H. Define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z), \quad k \in H. \tag{13}$$

Define a bilinear form on $L^2(E; m)$ by

$$\begin{aligned} \mathcal{E}(u, v) &= \int \Gamma(\nabla u(z), \nabla v(z))\, m(dz) \\ D(\mathcal{E}) &= \mathcal{F}C_b^\infty, \end{aligned} \tag{14}$$

where $\Gamma(\,\cdot, \cdot\,)$ is a positive semi-definite, continuous, bilinear form on H, which satisfies the sector condition. A common example is to let $\Gamma(\,\cdot, \cdot\,) = \langle\,\cdot, \cdot\,\rangle_H$, but in general Γ is not assumed to be symmetric. Then $\mathcal{E}$ is also a densely defined, positive semi-definite, bilinear form on $\mathcal{F}C_b^\infty$, satisfying the sector condition, because $\mathcal{E}$ inherits those properties from Γ. If such an $\mathcal{E}$ is closable in $L^2(E; m)$, then its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form.

Proposition 18. *The form $(\mathcal{E}, D(\mathcal{E}))$, defined as the closure of $(\mathcal{E}, \mathcal{F}C_b^\infty)$ above, is quasi-regular.*

The bulk of the proof is dedicated to showing (QR1), which requires coming up with compact sets that exhaust the space in the appropriate manner. It eventually goes back to a special choice of functions in $D(\mathcal{E})$ which are shown to converge in $\tilde{\mathcal{E}}_1^{1/2}$-norm. An application of Lemma 12 gives a nest $(F_k)_{k\in\mathbb{N}}$ on which the convergence is uniform, and topological considerations are used to conclude that the members of this nest are in fact compact. The interested reader is referred to [RS 92] or [MR 92] for details.

This proof has also been generalized [RS 93] to “square-field operator"-type Dirichlet forms, which are similar to the one seen in equation (14). Such Dirichlet forms can be used to construct and study a variety of infinite dimensional processes; such as infinite dimensional Ornstein-Uhlenbeck processes ([S 90a], [S 93]), diffusions on loop space ([ALR 92], [DR 92], [G 89]), and certain Fleming-Viot (measure-valued) processes ([OvRS 93], [S 91]). Once the quasi-regularity of the form has been established and the associated process constructed, you can use the Dirichlet form to analyze the behaviour of the sample paths. Determining what parts of the state space the process visits and what parts it avoids, helps you to better understand the model. We conclude by presenting two concrete examples where a Dirichlet form is used to study an infinite dimensional process. These two examples are extracted from [S 93] and [OvRS 93] respectively, where the reader can find more details. I hope that these give a flavor of the type of results one can get using Dirichlet forms, and the kind of calculations required to get them.

Example 19. Walsh's stochastic model of neural response.

In his 1981 paper "A Stochastic Model of Neural Response," J. Walsh [W 81] proposed a model for a nerve cylinder undergoing random stimulation along its length. The cylinder itself is modelled using the interval $[0, L]$ while $\{X(x, t, \omega) : 0 \le x \le L,\ t \ge 0\}$ denotes the nerve membrane potential at the time t at a location x along the axis. He found that this potential could be approximated by the solution $\{X_t\}$ of the equation

$$dX_t = -(I - \partial^2/\partial x^2)X_t\,dt + dW_t. \tag{15}$$

The Laplacian $\partial^2/\partial x^2$ is given reflecting boundary conditions at the endpoints 0 and L, and W is a white noise based on the measure $\eta(x)dx\,dt$. We set

$$\begin{aligned} A &= I - \partial^2/\partial x^2, \\ H &= L^2([0, L]; dx/\eta(x)). \end{aligned} \tag{16}$$

The symmetric case $\eta \equiv 1$ serves as a benchmark, but we will work in a more general setting and in fact only assume that η is absolutely continuous and $\eta' \in L^2([0, L]; dx)$. We also assume that η is bounded away from zero on $[0, L]$, consequently H is just $L^2([0, L]; dx)$ equipped with a different norm.

Just to keep things straight we mention that unlabelled quantities like $\langle\,,\rangle$ and $\|\ \|$ are the usual inner product and norm on $L^2([0, L]; dx)$, while labelled versions $\langle\,,\rangle_H$ and $\|\ \|_H$ are the inner product and norm in the $L^2([0, L]; dx/\eta(x))$ sense. The eigenvectors of A are given by $e_0 \equiv L^{-1/2}$ and $e_j(x) = \sqrt{2}L^{-1/2}\cos(\pi j x L^{-1})$ for $j \ge 1$, with $Ae_j = \lambda_j e_j = (1 + \pi^2 j^2 L^{-2})e_j$. Now the solution of (15) is a Gaussian process, in fact, an Ornstein-Uhlenbeck process in infinite dimensions. The invariant measure m for this process is also Gaussian and satisfies

$$\int \langle h, z\rangle_H \langle k, z\rangle_H\, m(dz) = (h, k) := \int_0^\infty \langle e^{-tA^*}h, e^{-tA^*}k\rangle_H\, dt. \tag{17}$$

Since $e^{-tA^*}f = \eta e^{-tA}(f/\eta)$, the above equation tells us that

$$\begin{aligned} &\int \langle e_j, z\rangle^2\, m(dz) \\ &\quad= \int \langle \eta e_j, z\rangle_H^2\, m(dz) \\ &\quad= \int_0^\infty \int \eta(x)\big(e^{-tA}(e_j)(x)\big)^2\, dx\, dt \\ &\quad= \frac{1}{2\lambda_j}\int \eta(x) e_j^2(x)\, dx \\ &\quad\le \frac{c}{\lambda_j}. \end{aligned} \tag{18}$$

Summing over j gives $\int \|z\|^2 \, m(dz) < \infty$, which tells us that the invariant measure m lives on H.

The conditions on η imply that the bilinear form (A^*h, k) is continuous and satisfies $(A^*h, k) + (A^*k, h) = \langle h, k\rangle_H$. (recall (17) for the definition of the round bracket form $(\,,\,)$). We now define a form on $\mathcal{F}C_b^\infty$ as in (14) by

$$\mathcal{E}(u, v) = \int (A^* \nabla u, \nabla v) \, dm. \tag{19}$$

Here the gradient is measured in the H sense, so that the linear map $\langle h, \cdot\,\rangle_H$ has gradient h, but the linear map $\langle h, \cdot\,\rangle = \langle \eta\, h, \cdot\,\rangle_H$ has gradient $\eta\, h$. From Proposition 18 and Theorem 14 we can construct the associated process $(\Omega, \mathcal{F}, \{X_t\}_{t\geq 0}, (P_z)_{z\in H})$ on H, and this process solves (15), in the sense of [AR 91]. So now we have a function-valued process $\{X_t\}_{t\geq 0}$ which lives in $L^2([0, L]; dx)$. What kind of functions are they? What would a snapshot of the process look like? Walsh had already showed that the process, in fact, takes values in $C[0, L]$ and has continuous sample paths there. We shall show that the function $x \to X(t, x)$ is, nevertheless, quite rough, in the sense of having unbounded variation.

Lemma 20. *$P_m(x \to X(x, t)$ is of unbounded variation for all $t) = 1$.*

Proof. Properties of cosine series show us that if $f \in L^2[0, L]$ has bounded variation, then the sequence of numbers $(\langle f, e_k\rangle)_{k\in\mathbb{N}}$ is $O(1/k)$. A calculation similar to the one done in (18) shows that $(\langle e_k, \cdot\,\rangle)_{k\in\mathbb{N}}$ on $L^2(H; m)$ are mean-zero Gaussian random variables with covariance

$$\int \langle e_j, z\rangle\langle e_k, z\rangle \, m(dz) = \Big(\int \eta(x) e_j(x) e_k(x) \, dx\Big) / (\lambda_j + \lambda_k). \tag{20}$$

Notice that the variance of $\langle e_j, \cdot\,\rangle$ is of the order j^{-2}. In addition, the conditions on η mean that the random variables $(\langle e_k, \cdot\,\rangle)_{k\in\mathbb{N}}$ not too strongly correlated and as a consequence you can show that for fixed N there exist $0 < \alpha < 1$ and $c > 0$ so that

$$m\left(\sup_{j=1}^{n} |j\langle e_j, z\rangle| \leq N\right) \leq c\alpha^n. \tag{21}$$

Letting $n \to \infty$ gives us $m(\sup_{j=1}^\infty |j\langle e_j, z\rangle| < \infty) = 0$, and using the observation above about cosine series we get,

$$m(z : x \to z(x) \text{ is of bounded variation}) = 0. \tag{22}$$

This gives us the fixed time result that for every $t \geq 0$,

$$P_m(x \to X(x, t) \text{ is of bounded variation}) = 0, \tag{23}$$

and now we'd like to use the Dirichlet form $\mathcal{E}$ to show that this is true for all t, with probability one.

Set $u_n(z) = \big(\sup_{j=1}^n |j\langle e_j, z\rangle| \wedge N\big)$. Then $u_n(z) \in D(\mathcal{E})$ and $u_n(z) \uparrow u(z)$ pointwise everywhere on E, where the function u satisfies $m(u \equiv N) = 1$. Calculating gradients gives

$$\nabla u_n(z) = (\pm k)(\eta e_k)\ \chi\Bigg(|k\langle e_k, z\rangle| = \sup_{j=1}^n |j\langle e_j, z\rangle| \leq N\Bigg). \tag{24}$$

Here χ means indicator function, so that $\nabla u_n(z)$ is zero if the supremum of $|j\langle e_j, z\rangle|$ is bigger than N, otherwise we get plus or minus $k\eta e_k$ where the index k is the same one at which the supremum occurred. The upshot is we get a nice bound for the norm of the gradient, because at each point z the gradient has at most one non-zero coordinate, so

$$\|\nabla u_n(z)\|_H^2 \leq c\,k^2\ \chi\Bigg(|k\langle e_k, z\rangle| = \sup_{j=1}^n |j\langle e_j, z\rangle| \leq N\Bigg). \tag{25}$$

Therefore, from (21) and (25), we obtain

$$\begin{aligned}\mathcal{E}(u_n, u_n) &= 1/2 \int \|\nabla u_n(z)\|_H^2\, m(dz)\\ &\leq c\,n^2 m\Bigg(\sup_{j=1}^n |j\langle e_j, z\rangle| \leq N\Bigg)\\ &\leq c\,n^2 \alpha^n.\end{aligned} \tag{26}$$

Applying Lemma 12 and Proposition 15 we find that

$$P_m\big(t \to u(X_t) \text{ is continuous}\big) = P_m\big(u(X_t) = N \text{ for all } t\big) = 1, \tag{27}$$

and since this is true for all N we conclude that

$$\begin{aligned}&P_m\big(\langle X_t, e_k\rangle \neq O(1/k) \text{ for all } t\big)\\ &\quad = P_m(x \to X(x,t) \text{ is of unbounded variation for all } t\big) = 1.\end{aligned} \tag{28}$$

Example 21. The Fleming-Viot process.

Let S be a locally compact separable metric space and set $E := \mathcal{P}(S) =$ all probability measures on S. Consider the space of functions

$$\mathcal{F}C_b^\infty := \{F((f_1,\cdot),\ldots,(f_m,\cdot)) : m \in \mathbb{N},\ F \in C_b^\infty(\mathbb{R}^m),\ f_1,\ldots,f_m \in C_b(S)\}, \tag{29}$$

where the round brackets refer to integration on S; i.e., $(f,\mu) = \int_S f(x)\,\mu(dx)$. For every $x \in S$ and $u \in \mathcal{F}C_b^\infty$ define

$$\frac{\partial u}{\partial \epsilon_x}(\mu) = \frac{d}{ds} u(\mu + s\epsilon_x)|_{s=0}. \tag{30}$$

Consider the Fleming-Viot generator

$$Lu(\mu) = \frac{1}{2}\int_S\int_S \mu(dx)\Big(\epsilon_x(dy) - \mu(dy)\Big)\frac{\partial^2 u}{\partial\epsilon_x\partial\epsilon_y}(\mu) + \int_S \mu(dx) A\Bigg(\frac{\partial u}{\partial \epsilon_x}(\mu)\Bigg). \tag{31}$$

We will only consider bounded mutation operators A of the form $Af(x) = (\theta/2)\int_S \big(f(\xi) - f(x)\big)\,\nu_0(d\xi)$ where $\theta > 0$ and $\nu_0 \in \mathcal{P}(S)$. S.N. Ethier and T.G. Kurtz [EK 93] have shown that there is a probability measure m on E such that

$$\mathcal{E}(u,v) = \int (-Lu)v\,dm \qquad u,v \in \mathcal{F}C_b^\infty, \tag{32}$$

defines a symmetric bilinear form. This form is closable on $L^2(E;m)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form. This form can be re-written in a more symmetric way on $\mathcal{F}C_b^\infty$ at least as:

$$\mathcal{E}(u,v) = \int \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu\, m(d\mu), \tag{33}$$

where $\langle f,g\rangle_\mu := \int fg\,d\mu - (\int f d\mu)(\int g d\mu)$ and, for each fixed μ, as a function of $x \in S$ we let $(\nabla u(\mu))(x) = (\partial u/\partial\epsilon_x)(\mu)$. The invariant measure m has the following form:

$$m(\cdot) = P\big(\sum_{i=1}^\infty \rho_i \epsilon_{\xi_i} \in \cdot\,\big). \tag{34}$$

Here $(\rho_1, \rho_2, \ldots)$ has a Poisson-Dirichlet distribution with parameter θ, and $\{\xi_i\}$ are i.i.d. S-valued random variables that are independent of $(\rho_1, \rho_2, \ldots)$ and have distribution ν_0.

Consider a linear function $u \in \mathcal{F}C_b^\infty$ given by $u(\mu) := \int_S f(x)\,\mu(dx)$, for some $f \in C_b(S)$. The average value of u is

$$\begin{aligned}\int_E u(\mu)\,m(\mu) &= E\big(\int_S f\,d(\Sigma\rho_i\epsilon_{\xi_i})\big)\\ &= \sum E(\rho_i)Ef(\xi_i)\\ &= \int_S f(x)\,\nu_0(dx).\end{aligned} \tag{35}$$

Taking partial derivatives we discover $(\partial u/\partial\epsilon_x)(\mu) = f(x)$ at every point $\mu \in E$, and therefore $\|\partial u/\partial\epsilon_x\|_\mu^2 = \int f^2\,d\mu - (\int f\,d\mu)^2$. Plugging this into the Dirichlet form we discover that

$$\mathcal{E}(u,u) = \int_E \big(\int f^2\,d\mu - (\int f\,d\mu)^2\big)\,m(d\mu). \tag{36}$$

Using (36) and the fact that $(\mathcal{E}, D(\mathcal{E}))$ is a closed form we pass may from continuous functions f to bounded, measurable f. In particular, for any fixed Borel set $F \subseteq S$, the map $\mu \to \mu(F)$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. Thus, if $\nu_0(F) = 0$, then $\mu(F) = 0$ m-a.e. from (35), and so $\mu(F) = 0$ $\mathcal{E}$-quasi-everywhere. We conclude that if $\nu_0(F) = 0$, then $P_m(X_t(F) = 0 \text{ for all } t) = 1$. In other words, if ν_0 fails to charge the set F, then the Fleming-Viot process will never charge it either, not even at exceptional times. More calculations in a similar vein lead to the fact that the function $u(\mu) = \sum(\text{atoms of } \mu)$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. But (34) shows that $u(\mu) = 1$ m-a.e., and so $u(\mu) = 1$ $\mathcal{E}$-quasi-everywhere, and therefore $P_m(X_t$ is purely atomic for all $t) = 1$. Thus we can recover, using Dirichlet forms, a result [EK 86; Chapter 10, Theorem 4.5] proved by Ethier and Kurtz. We can also use $\mathcal{E}$ to discover other properties of this measure-valued process ([OvRS 93]).

References

[AH-K 75] S. Albeverio and R. Høegh-Krohn: Quasi-invariant measures, symmetric diffusion processes and quantum fields. In: Les Méthodes Mathématiques de la Théorie Quantique des Champs, Colloques Internationaux du C.N.R.S., no. **248**, Marseille, 23-27 juin 1975, C.R.N.S., 1976.

[AH-K 77a] S. Albeverio and R. Høegh-Krohn: Dirichlet forms and diffusion processes on rigged Hilbert spaces. Z. Wahrsch. verw. Geb. **40**, 1-57 (1977).

[AH-K 77b] S. Albeverio and R. Høegh-Krohn: Hunt processes and analytic potential theory on rigged Hilbert spaces. Ann. Inst. Henri Poincaré Sect. B **13**, 269-291 (1977).

[ALR 92] S. Albeverio, R. Léandre, and M. Röckner: Construction of a rotational invariant diffusion on the free loop space. Preprint.

[AM 91a] S. Albeverio and Z.M. Ma: Diffusion processes associated with singular Dirichlet forms. In: Proc. Stochastic Analysis and Applications. ed. A.B. Cruzeiro and J.C. Zambrini. New York: Birkhäuser 1991.

[AM 91b] S. Albeverio and Z.M. Ma: Necessary and sufficient conditions for the existence of m-perfect processes associated with Dirichlet forms. In: Séminaire de Probabilités XXV. Lecture Notes in Mathematics **1485**, 374-406, Berlin: Springer 1991.

[AMR 93a] S. Albeverio, Z.M. Ma, and M. Röckner: Local property of Dirichlet forms and diffusions on general state spaces. Math. Ann. **296**, 677-686 (1993).

[AMR 93b] S. Albeverio, Z.M. Ma, and M. Röckner: Quasi-regular Dirichlet forms and Markov processes. J. Funct. Anal. **111**, 118-154 (1993).

[AR 89] S. Albeverio and M. Röckner: Classical Dirichlet forms on topological vector spaces - the construction of the associated diffusion process. Probab. Th. Rel. Fields **83**, 405-434 (1989).

[AR 90] S. Albeverio and M. Röckner: Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula. J. Funct. Anal. **88**, 395-436 (1990).

[AR 91] S. Albeverio and M. Röckner: Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. Probab. Th. Rel. Fields. **89**, 347-386 (1991).

[BH 91] N. Bouleau and F. Hirsch: Dirichlet Forms and Analysis on Wiener Space. Berlin · New York: Walter de Gruyter 1991.

[CaMe 75] S. Carrillo Menendez: Processus de Markov associé a une forme de Dirichlet non symétrique. Z. Wahrsch. verw. Geb. **33**, 139-154 (1975).

[DR 92] B. Driver and M. Röckner: Construction of diffusions on path and loop spaces of compact Riemannian manifolds. Preprint (1992).

[EK 86] S.N. Ethier and T.G. Kurtz: Markov Processes - Characterization and Convergence. New York: John Wiley & Sons 1986.

[EK 93] S.N. Ethier and T.G. Kurtz: Convergence to Fleming-Viot processes in the weak atomic topology. To appear in Stochastic Processes Appl. (1993).

[FdL 78] D. Feyel and A. de La Pradelle: Dualité des quasi-résolvantes de Ray. In: Sem. Th. Potentiel Paris no. 4. Lecture Notes in Mathematics **713**, 67-88, Berlin: Springer 1979.

[F 80] M. Fukushima: Dirichlet Forms and Markov Processes. Amsterdam: North Holland 1980.

[G 89] E. Getzler: Dirichlet forms on loop space. Bull. Sci. Math. (2) **113**, 151-174 (1989).

[Ku 82] S. Kusuoka: Dirichlet forms and diffusion processes on Banach spaces. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29**, 79-95 (1982).

[Le 82] Y. LeJan: Dual Markovian semigroups and processes. In: Functional Analysis in Markov Processes. ed. M. Fukushima. Lecture Notes in Mathematics **923**, 47-75, Berlin: Springer 1982.

[LR 92] T. Lyons and M. Röckner: A note on tightness of capacities associated with Dirichlet forms. Bull. London Math. Soc. **24**, 181-184 (1992).

[MR 92] Z.M. Ma and M. Röckner: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Berlin: Springer 1992.

[O 88] Y. Oshima: Lectures on Dirichlet Forms. Preprint Erlangen (1988).

[OvR 93] L. Overbeck and M. Röckner: Existence of Markov processes associated with Semi-Dirichlet forms. Preprint (1993). Publication in preparation.

[OvRS 93] L. Overbeck, M. Röckner, and B. Schmuland: Dirichlet forms and Fleming-Viot processes. Publication in preparation.

[RS 92] M. Röckner and B. Schmuland: Tightness of general $C_{1,p}$ capacities on Banach space. J. Funct. Anal. **108**, 1-12 (1992).

[RS 93] M. Röckner and B. Schmuland: Quasi-regular Dirichlet forms: Examples and counterexamples. Preprint (1993). Publication in preparation.

[RZ 92] M. Röckner and T.S. Zhang: Uniqueness of generalized Schrödinger operators and applications. J. Funct. Anal. **105**, 187-231 (1992).

[S 90a] B. Schmuland: Sample path properties of ℓ^p-valued Ornstein-Uhlenbeck processes. Canadian Math. Bulletin **33(3)**, 358-366 (1990).

[S 90b] B. Schmuland: An alternative compactification for classical Dirichlet forms on topological vector spaces. Stochastics and Stochastic Reports **33**, 75-90 (1990).

[S 91] B. Schmuland: A result on the infinitely many neutral alleles diffusion model. J. Appl. Probab. **28**, 253-267 (1991).

[S 92] B. Schmuland: Tightness of Gaussian capacities on subspaces. C. R. Math. Rep. Acad. Sci. Canada **14**, 125-130 (1992).

[S 93] B. Schmuland: Non-symmetric Ornstein-Uhlenbeck processes in Banach space via Dirichlet forms. To appear in the Canadian Journal of Mathematics.

[Sh 88] M. Sharpe: General Theory of Markov Processes. New York: Academic Press 1988.

[W 81] J.B. Walsh: A stochastic model of neural response. Adv. Appl. Probab. **14**, 231-281 (1981).