The law of large numbers and the law of the iterated logarithm for infinite dimensional interacting diffusion processes

Byron Schmuland and Wei Sun

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Abstract

The classical Dirichlet form given by the intrinsic gradient on $\Gamma_{\mathbb{R}^d}$ is associated with a Markov process consisting of a countable family of interacting diffusions. By considering each diffusion as a particle with unit mass, the randomly evolving configuration can be thought of as a Radon measure valued diffusion.

The quasi-sure analysis of Dirichlet forms is used to find exceptional sets of configurations for this Markov process. We consider large scale properties of the configuration and show that, for quite general measures, the process never hits those unusual configurations that violate the law of large numbers. Furthermore, for certain Gibbs measures, which model random particles in $\mathbb{R}^d$ that interact via a potential function, we show, for $d \leq 3$, that the process never hits those unusual configurations that violate the law of the iterated logarithm.

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1 Introduction

In a pair [3, 4] of fundamental papers in 1998, Albeverio, Kondratiev, and Röckner began a study of analysis and geometry on configuration space.

(This work had been previously announced in [1, 2] and anticipated in the 1996 papers by Osada [13] and Yoshida [25].) One of the features of [3, 4] was the construction of a configuration-valued Markov process using a Dirichlet form based on an intrinsic gradient defined on configuration space.

In this paper, we look at some sample path properties of the Albeverio-Kondratiev-Röckner process in the case where the underlying manifold is d -dimensional Euclidean space. For quite general stationary measures for the process, we will show that the associated Markov process never hits configurations that violate the law of large numbers. Furthermore, for a Ruelle measure μ with small activity parameter z the law of the iterated logarithm (Proposition 4) holds for μ -almost every configuration $\gamma \in \Gamma_{\mathbb{R}^d}$. In dimensions d less than or equal to 3, we also show that the associated Markov process never hits configurations that violate the law of the iterated logarithm. We do so by proving that these sets of unusual configurations are exceptional sets for the Dirichlet form.

2 Classical Dirichlet forms on configuration space

In this section we recall some of the basic definitions and properties about the classical Dirichlet forms on configuration spaces. For more detailed definitions and fuller explanations we refer the reader to [3, 4, 9, 16].

The space of locally finite configurations in $\mathbb{R}^d$ is defined by

$$\Gamma_{\mathbb{R}^d} := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for every compact } K\}.$$

A configuration γ will be identified with the Radon measure $\sum_{x \in \gamma} \varepsilon_x$. The space $\Gamma_{\mathbb{R}^d}$ will be given the topology of vague convergence of measures, and measures on $\Gamma_{\mathbb{R}^d}$ are defined on the corresponding Borel sets $\mathcal{B}(\Gamma_{\mathbb{R}^d})$.

For $f \in C_0(\mathbb{R}^d)$ we let $\langle f, \gamma \rangle$ be the integral of f with respect to the measure γ , that is, $\langle f, \gamma \rangle = \sum_{x \in \gamma} f(x)$. Define

$$\begin{aligned} \mathcal{FC}_b^\infty := \{ u : u(\gamma) = g(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \dots, \langle f_n, \gamma \rangle) \\ \text{for some } f_i \in C_0^\infty(\mathbb{R}^d) \text{ and } g \in C_b^\infty(\mathbb{R}^n) \}. \end{aligned}$$

For $u \in \mathcal{FC}_b^\infty$, we define the gradient $\nabla^\Gamma u$ at the point $\gamma \in \Gamma_{\mathbb{R}^d}$ as an

element of the “tangent space” $T_\gamma(\Gamma_{\mathbb{R}^d}) := L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \gamma)$ by the formula

$$(\nabla^\Gamma u)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \dots, \langle f_n, \gamma \rangle) \nabla f_i(x).$$

Here ∇ refers to the usual gradient on $\mathbb{R}^d$. It is not hard to prove that $\nabla^\Gamma u$ is well-defined, even though the representation of u as a cylinder function is not unique.

Definition 1 For $u, v \in \mathcal{FC}_b^\infty$ define the square field $\mathbb{H}(u, v)^1$ as the real-valued function on $\Gamma_{\mathbb{R}^d}$ given by

$$\begin{aligned} \mathbb{H}(u, v)(\gamma) &:= \langle \nabla^\Gamma u, \nabla^\Gamma v \rangle_{T_\gamma(\Gamma_{\mathbb{R}^d})} \\ &= \int_{\mathbb{R}^d} \langle (\nabla^\Gamma u)(\gamma; x), (\nabla^\Gamma v)(\gamma; x) \rangle_{\mathbb{R}^d} \gamma(dx). \end{aligned}$$

We will often use the abbreviation $\mathbb{H}(u) := \mathbb{H}(u, u)$.

In this section we fix a probability measure μ on $\Gamma_{\mathbb{R}^d}$ with Radon mean, that is,

$$\int_{\Gamma_{\mathbb{R}^d}} \gamma(K) \mu(d\gamma) < \infty \text{ for all compact } K \subset \mathbb{R}^d. \quad (\mu.1)$$

Furthermore, assume that

$$\nabla^\Gamma u = \nabla^\Gamma v \text{ } \mu\text{-a.e. if } u, v \in \mathcal{FC}_b^\infty \text{ such that } u = v \text{ } \mu\text{-a.e.} \quad (\mu.2)$$

Definition 2 For $u, v \in \mathcal{FC}_b^\infty$ define the pre-Dirichlet form by

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\Gamma_{\mathbb{R}^d}} \mathbb{H}(u, v)(\gamma) \mu(d\gamma).$$

Let $\mathcal{F}_b^{\infty, \mu}$ denote the set of μ -equivalence classes determined by $\mathcal{FC}_b^\infty$. Suppose that $(\mathcal{E}, \mathcal{F}_b^{\infty, \mu})$ is closable on $L^2(\Gamma_{\mathbb{R}^d}; \mu)$ and denote the closure by $(\mathcal{E}, D(\mathcal{E}))$. We refer to [3, 4, 9, 16] for concrete examples. By [9], $(\mathcal{E}, D(\mathcal{E}))$ is then a symmetric, quasi-regular and local Dirichlet form. The quasi-regularity and locality of $(\mathcal{E}, D(\mathcal{E}))$ has been proven for certain cases by Yoshida [25], and in general by Ma and Röckner [9] but since $\Gamma_{\mathbb{R}^d}$ is not

¹This notation is based on the Chinese character Tián, which means ‘field’.

complete with respect to the vague topology it is necessary to use the completed state space

$$\ddot{\Gamma}_{\mathbb{R}^d} := \{\mathbb{Z}_+ \cup \{+\infty\}\text{-valued Radon measures on } \mathbb{R}^d\}.$$

Since $\Gamma_{\mathbb{R}^d} \subset \ddot{\Gamma}_{\mathbb{R}^d}$ and $\mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d}) \cap \Gamma_{\mathbb{R}^d} = \mathcal{B}(\Gamma_{\mathbb{R}^d})$, we can consider μ as a measure on $(\ddot{\Gamma}_{\mathbb{R}^d}, \mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma}_{\mathbb{R}^d}; \mu)$. The associated strong Markov process $(X_t)_{t \geq 0}$ has vaguely continuous sample paths since $(\mathcal{E}, D(\mathcal{E}))$ is a local form [8, Chapter V, Theorem 1.11]. Note that by [19, 10], if $d \geq 2$, the set $\ddot{\Gamma}_{\mathbb{R}^d} \setminus \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$ -exceptional for a large class of measures μ on $\Gamma_{\mathbb{R}^d}$.

We recall the following results from Dirichlet form theory. Lemma 1 (see [23]) is used to prove that certain sets are $\mathcal{E}$ -exceptional while Lemma 2 gives us the interpretation in terms of the sample paths of $(X_t)_{t \geq 0}$.

Lemma 1 *Let $u_n \in D(\mathcal{E})$ be a sequence of $\mathcal{E}$ -quasi-continuous functions with $\sup_n \mathcal{E}(u_n, u_n) < \infty$ and $u_n \rightarrow u$ pointwise. Then u is an $\mathcal{E}$ -quasi-continuous function, in particular, for μ -almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$,*

$$P_\gamma(t \rightarrow u(X_t) \text{ is continuous}) = 1.$$

If u is μ -square integrable, then $u \in D(\mathcal{E})$.

Lemma 2 *A set $N \in \mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d})$ is $\mathcal{E}$ -exceptional if and only if, for μ -almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$,*

$$P_\gamma(X_t \in N \text{ for some } 0 \leq t < \infty) = 0.$$

We sometimes refer to proofs of exceptionality as *capacitary* since N is $\mathcal{E}$ -exceptional if and only if $\text{Cap}(N) = 0$ for a suitably defined capacity Cap on $\ddot{\Gamma}_{\mathbb{R}^d}$ [8, Chapter III, Theorem 2.11].

3 Law of large numbers on configuration space

Define an Abelian group $(T_r)_{r \in \mathbb{Z}^d}$ of automorphisms on $\ddot{\Gamma}_{\mathbb{R}^d}$ by $T_r \gamma(G) = \gamma(G - r)$ for any bounded Borel G . In this section we fix a probability measure μ on $\ddot{\Gamma}_{\mathbb{R}^d}$ which is invariant with respect to $(T_r)_{r \in \mathbb{Z}^d}$ and satisfies the conditions $(\mu.1)$ and $(\mu.2)$ in Section 2.

For $l > 0$ let C_l be the cube $[-l, l]^d$, and for every $r \in \mathbb{Z}^d$ let $Q_r = r + [0, 1]^d$. Denote the Lebesgue measure of any Borel subset G of $\mathbb{R}^d$ by $|G|$. Define

$$LLN := \left\{ \gamma \in \ddot{\Gamma}_{\mathbb{R}^d} : \lim_{n \in \mathbb{N}} \frac{\gamma(C_n)}{|C_n|} \text{ exists} \right\} = \left\{ \gamma \in \ddot{\Gamma}_{\mathbb{R}^d} : \lim_{l \in \mathbb{R}_+} \frac{\gamma(C_l)}{|C_l|} \text{ exists} \right\}.$$

We first recall in Proposition 1 the well-known fixed time law of large numbers ([11, Proposition 4.23]), while Proposition 2 gives the full capacity version.

Proposition 1 *The law of large numbers holds for μ -almost every γ , that is, $\mu(LLN^c) = 0$.*

Proof. The family $\{\gamma(G) : G \in \mathcal{B}(\mathbb{R}^d), \text{ bounded}\}$ is trivially an additive covariant spatial process in the sense of [11]. Therefore, by [11, Proposition 4.23],

$$\lim_{n \rightarrow \infty} \frac{\gamma(C_n)}{|C_n|} = E_\mu(\gamma(Q_0) \mid \mathcal{H}), \quad \mu\text{-a.e.},$$

where $\mathcal{H}$ denotes the σ -algebra of $(T_r)_{r \in \mathbb{Z}^d}$ -invariant sets in $\mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d})$. $\square$

Proposition 2 *The set LLN^c is $\mathcal{E}$ -exceptional.*

Proof. For every $n \geq 3$, let ψ_n be a smooth function satisfying $1_{[-n+2, n-2]} \leq \psi_n \leq 1_{[-n, n]}$, and $|\psi'_n| \leq 1$. Define a continuous element of $D(\mathcal{E})$ by

$$u_n(\gamma) := \langle \prod_{i=1}^d \psi_n(x_i), \gamma \rangle / |C_n|.$$

Taking the limsup through the inequality

$$\frac{|C_{n-2}|}{|C_n|} \frac{\gamma(C_{n-2})}{|C_{n-2}|} \leq u_n(\gamma) \leq \frac{\gamma(C_n)}{|C_n|},$$

we find that $u(\gamma) := \limsup_n u_n(\gamma) = \limsup_n \gamma(C_n)/|C_n|$.

Bounding the square field gives

$$\boxplus(u_n)(\gamma) \leq d \frac{\gamma(C_n)}{|C_n|^2}. \tag{1}$$

Since C_n is an increasing sequence, there exists a sequence $(r_m)_{m \in \mathbb{N}}$ in $\mathbb{Z}^d$ so $C_n = \cup_{m=1}^{|C_n|} Q_{r_m}$ for $n \in \mathbb{N}$. Then

$$\frac{\chi_{C_n}}{|C_n|^2} \leq \sum_{m=1}^{|C_n|} \frac{\chi_{Q_{r_m}}}{m^2} \leq \sum_{m=1}^{\infty} \frac{\chi_{Q_{r_m}}}{m^2},$$

so that

$$\begin{aligned} \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{n \in \mathbb{N}} \frac{\gamma(C_n)}{|C_n|^2} \mu(d\gamma) &= \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} \frac{\chi_{C_n}(x)}{|C_n|^2} \gamma(dx) \mu(d\gamma) \\ &\leq \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \int_{\mathbb{R}^d} \sup_{n \in \mathbb{N}} \frac{\chi_{C_n}(x)}{|C_n|^2} \gamma(dx) \mu(d\gamma) \\ &\leq \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \int_{\mathbb{R}^d} \sum_{m=1}^{\infty} \frac{\chi_{Q_{r_m}}(x)}{m^2} \gamma(dx) \mu(d\gamma) \\ &= \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sum_{m=1}^{\infty} \frac{\gamma(Q_{r_m})}{m^2} \mu(d\gamma) \\ &= \sum_{m=1}^{\infty} \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \frac{\gamma(Q_{r_m})}{m^2} \mu(d\gamma) \\ &= \sum_{m=1}^{\infty} \frac{\mu(Q_0)}{m^2} \\ &< \infty. \end{aligned}$$

Let us denote the random variable $X^*(\gamma) := \sup_{n \in \mathbb{N}} \gamma(C_n)/|C_n|^2$. For fixed $n \in \mathbb{N}$ let $A_j = \{n, n+1, \dots, n+j\}$ so that

$$\sup_{k \geq n} u_k(\gamma) = \sup_j \sup_{k \in A_j} u_k(\gamma).$$

Now for each $j \in \mathbb{N}$, $\sup_{k \in A_j} u_k \in D(\mathcal{E})$ and is $\mathcal{E}$ -quasi-continuous. Repeated use of the inequality $\boxplus(u \vee v) \leq \boxplus(u) \vee \boxplus(v)$, for $u, v \in D(\mathcal{E})$ combined with the bound (1) gives $\boxplus(\sup_{k \in A_j} u_k) \leq d X^*$, and so

$$\sup_j \mathcal{E} \left(\sup_{k \in A_j} u_k, \sup_{k \in A_j} u_k \right) \leq d \int_{\ddot{\Gamma}_{\mathbb{R}^d}} X^*(\gamma) \mu(d\gamma) < \infty.$$

Applying Lemma 1, we see that the pointwise limit $\sup_{k \geq n} u_k$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$ -quasi-continuous. In addition, the bound for the square

field also carries over; $\boxplus(\sup_{k \geq n} u_k) \leq dX^*$. Applying the same argument to the decreasing sequence $(\sup_{k \geq n} u_k)_{n \in \mathbb{N}}$, we find that the pointwise limit u belongs to $D(\mathcal{E})$ and is $\mathcal{E}$ -quasi-continuous.

A parallel argument shows that $v(\gamma) := \liminf_{n \rightarrow \infty} \gamma(C_n)/|C_n|$ is also $\mathcal{E}$ -quasi-continuous. Since the two $\mathcal{E}$ -quasi-continuous functions u and v agree μ -almost everywhere, they must agree except on an $\mathcal{E}$ -exceptional set [8, Chapter IV, Proposition 3.3]. $\square$

Remark. In the above argument we only use the translation invariant property of μ to ensure that $\sup_{r \in \mathbb{Z}^d} E_\mu(\gamma(Q_r)) < \infty$. So one can expect that the capacitary version of the law of large numbers also holds for more general measures only if the fixed time law of large numbers holds. In particular, this argument also gives the capacitary version of the law of large numbers for (mixed) Poisson point processes in $\mathbb{R}^d$. This improves the result [22, Proposition 5] by removing the assumption that $\int_{\mathbb{R}_+} z \log^+(z) \lambda(dz) < \infty$.

4 Gibbs measures on configuration space

In this section we give some of the preliminaries on Gibbs measure we will need to prove our results. Let σ be a measure on $\mathbb{R}^d$ that has a density ρ with respect to Lebesgue measure satisfying $\rho > 0$ almost everywhere, and $\rho^{1/2} \in H_{loc}^{1,2}(\mathbb{R}^d)$. Here $H_{loc}^{1,2}(\mathbb{R}^d)$ denotes the local Sobolev space of order 1 in $L_{loc}^2(\mathbb{R}^d; m)$. The *Poisson measure* π_σ with intensity measure σ is the probability measure on $\Gamma_{\mathbb{R}^d}$ characterized by:

$$\int_{\Gamma_{\mathbb{R}^d}} \exp(\langle f, \gamma \rangle) \pi_\sigma(d\gamma) = \exp \left(\int_{\mathbb{R}^d} (e^{f(x)} - 1) \sigma(dx) \right),$$

for $f \in C_0(\mathbb{R}^d)$. A *mixed Poisson measure* is given by:

$$\mu := \int_{\mathbb{R}_+} \pi_{z\sigma} \lambda(dz),$$

where λ is a probability measure on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} z \lambda(dz) < \infty$.

A *pair potential* is any measurable function $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x)$. For a pair potential ϕ , a bounded measurable subset

Λ in $\mathbb{R}^d$, and a configuration $\gamma \in \Gamma_{\mathbb{R}^d}$, the *conditional energy* of γ in Λ is given by the formula

$$E_{\Lambda}^{\phi}(\gamma) := \begin{cases} \sum \phi(x-y) & \text{if } \sum |\phi(x-y)| < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

where the summation is taken over all pairs $\{x, y\} \subset \gamma$ such that $\{x, y\} \cap \Lambda \neq \emptyset$. We adopt the convention that a sum over the empty set is zero so that $E_{\Lambda}^{\phi}(\gamma) = 0$ if either $\gamma(\mathbb{R}^d) = 1$ or $\gamma(\Lambda) = 0$. We also define

$$Z_{\Lambda}^{z, \phi}(\gamma) := \int_{\Gamma_{\mathbb{R}^d}} \exp \left[-E_{\Lambda}^{\phi}(\gamma_{\Lambda^c} + \omega_{\Lambda}) \right] \pi_{z\sigma}(d\omega).$$

Here $\gamma_{\Lambda^c} + \omega_{\Lambda}$ is the configuration formed by combining the part of γ outside Λ with the part of ω inside Λ . The parameter $z > 0$ is called the *activity*.

Definition 3 A probability measure μ on $\Gamma_{\mathbb{R}^d}$ is called a Gibbs measure with activity z , pair potential ϕ , and intensity measure σ if, for every bounded measurable $\Lambda \subset \mathbb{R}^d$ we have $Z_{\Lambda}^{z, \phi}(\gamma) < \infty$ for μ -almost every $\gamma \in \Gamma_{\mathbb{R}^d}$ and for every $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$,

$$\mu(\Delta) = \iint_{\Gamma_{\mathbb{R}^d} \Gamma_{\mathbb{R}^d}} 1_{\Delta}(\gamma_{\Lambda^c} + \omega_{\Lambda}) \frac{\exp \left[-E_{\Lambda}^{\phi}(\gamma_{\Lambda^c} + \omega_{\Lambda}) \right]}{Z_{\Lambda}^{z, \phi}(\gamma)} \pi_{z\sigma}(d\omega) \mu(d\gamma). \quad (2)$$

We recall the definition of the cubes $Q_r = r + [0, 1]^d$ for $r \in \mathbb{Z}^d$.

Definition 4 (SS) A pair potential ϕ is called superstable if there exist $A > 0$ and $B \geq 0$ so that if $\Lambda = \cup_{r \in R} Q_r$ is a finite union of cubes, then

$$E_{\Lambda}^{\phi}(\gamma_{\Lambda}) \geq \sum_{r \in R} [A\gamma(Q_r)^2 - B\gamma(Q_r)].$$

(LR) A pair potential ϕ is called lower regular if there exists a decreasing positive function $\Psi : \mathbb{N} \rightarrow [0, \infty)$ such that $\sum_{r \in \mathbb{Z}^d} \Psi(|r|_{\infty}) < \infty$, and for any disjoint Λ' and Λ'' that are finite unions of cubes, then we have

$$\iint_{\Lambda' \Lambda''} \phi(x-y) \gamma(dx) \gamma(dy) \geq - \sum_{r', r'' \in \mathbb{Z}^d} \Psi(|r' - r''|_{\infty}) \gamma_{\Lambda'}(Q_{r'}) \gamma_{\Lambda''}(Q_{r''}),$$

for all $\gamma \in \Gamma_{\mathbb{R}^d}$. Here $|\cdot|_{\infty}$ refers to the maximum norm on $\mathbb{R}^d$.

(I) A pair potential ϕ is called integrable if $\int_{\mathbb{R}^d} |\exp(-\phi(x)) - 1| dx < \infty$.

Definition 5 A measure μ on $\Gamma_{\mathbb{R}^d}$ is called tempered if

$$\limsup_{l \rightarrow \infty} \frac{\sum_{|r| \leq l} \gamma(Q_r)^2}{(2l+1)^d} < \infty \text{ for } \mu\text{-almost every } \gamma \in \Gamma_{\mathbb{R}^d}.$$

Definition 6 A probability measure μ on $\Gamma_{\mathbb{R}^d}$ is called a Ruelle measure if μ is a tempered Gibbs measure with activity parameter $z > 0$, intensity σ equal to Lebesgue measure, and a pair potential ϕ that is superstable, lower regular, and integrable.

Suppose that μ is a Gibbs measure and Λ a bounded measurable subset of $\mathbb{R}^d$. Let $\mathcal{F}(\Lambda)$ be the σ -algebra of events $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$ that only depend on the part of the configuration in Λ , that is, $1_\Delta(\gamma) = 1_\Delta(\gamma_\Lambda)$ for every $\gamma \in \Gamma_{\mathbb{R}^d}$. Exchanging the order of integration in (2), we find that $\mu|_{\mathcal{F}(\Lambda)}$ is absolutely continuous with respect to $\pi_{z\sigma}|_{\mathcal{F}(\Lambda)}$ with density

$$\omega \mapsto \int_{\Gamma_{\mathbb{R}^d}} [Z_\Lambda^{z,\phi}(\gamma)]^{-1} \exp \left[-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda) \right] \mu(d\gamma). \quad (3)$$

In other words, a Gibbs measure is always locally absolutely continuous with respect to its corresponding Poisson measure. In general we have very little information about the density (3), but for Ruelle measures it is known to be bounded, with a bound that depends on $|\Lambda|$, the Lebesgue measure of Λ . In particular, for any $n \in \mathbb{N}$, the moments satisfy

$$E_\mu(\gamma(\Lambda)^n) \leq c(n, |\Lambda|). \quad (4)$$

Another important tool in studying Ruelle measures is the family of (infinite-volume) correlation functions $\rho_m : (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \rho_m(x_1, \dots, x_m) \\ = z^m \exp\left(-\sum_{i < j} \phi(x_i - x_j)\right) \int_{\Gamma_{\mathbb{R}^d}} \exp\left(-\sum_{i=1}^m \langle \phi(x_i - \cdot), \gamma \rangle\right) \mu(d\gamma). \end{aligned}$$

These provide us with useful formulas for Ruelle measures:

$$\int_{\Gamma_{\mathbb{R}^d}} \langle f, \gamma \rangle \mu(d\gamma) = \int_{\mathbb{R}^d} f(x) \rho_1(x) dx, \quad (5)$$

$$\int_{\Gamma_{\mathbb{R}^d}} (\langle f, \gamma \rangle \langle g, \gamma \rangle - \langle fg, \gamma \rangle) \mu(d\gamma) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(y) \rho_2(x, y) dx dy. \quad (6)$$

From now on we will simply assume that the pair potential ϕ satisfies the (mild) additional smoothness and integrability assumptions to ensure that μ satisfies the conditions in Section 2. By [16, Theorem 6.13 and Remark 7.5] (cf. also [4, Proposition 5.1]) and [9], the classical Dirichlet form given by integrating the gradient on $\Gamma_{\mathbb{R}^d}$ against such a Ruelle measure is a symmetric, quasi-regular and local Dirichlet form.

For sufficiently small z , by [21, Theorems 5.7 and 5.8], μ is the unique Gibbs measure for ϕ and z , and is translation invariant. In particular, the first correlation function $\rho_1(x) = \rho$ is constant.

Proposition 3 *For sufficiently small z , the complement of the set*

$$\left\{ \gamma \in \ddot{\Gamma}_{\mathbb{R}^d} : \lim_{n \rightarrow \infty} \frac{\gamma(C_n)}{|C_n|} = \rho \right\}$$

is $\mathcal{E}$ -exceptional.

Proof. Since μ is the unique Gibbs measure, it is translation-ergodic by [15, Theorem 4.1], so

$$E_\mu(\gamma(Q_0) \mid \mathcal{H}) = E_\mu(\gamma(Q_0)) = \rho, \quad \mu\text{-a.e.}$$

which gives the result using Proposition 1 and Proposition 2.

5 Law of the iterated logarithm on configuration space

In this section we prove a capacitary version of the law of the iterated logarithm. This time let $V_n = [-n^{1/d}/2, n^{1/d}/2]^d$ be the cube with volume n , and set $\mu_n = E_\mu(\gamma(V_n))$, $\sigma_n^2 = \text{Var}_\mu(\gamma(V_n))$, and $\chi_n := (2\sigma_n^2 \log \log \sigma_n^2)^{1/2}$. Letting $S_n = \gamma(V_n) - \mu_n$, we define

$$LIL := \left\{ \gamma \in \ddot{\Gamma}_{\mathbb{R}^d} : \limsup_n \frac{S_n}{\chi_n} = 1 \text{ and } \liminf_n \frac{S_n}{\chi_n} = -1 \right\}.$$

As with the law of large numbers, we must begin by proving the fixed time result, that is, $\mu(LIL^c) = 0$. The proof is based on the methods

used in [12, 26], while replacing the strong mixing condition there by the exponential mixing condition in Lemma 3 below.

Even this fixed time result is, as far as we know, the first law of the iterated logarithm for Gibbsian random fields so we've included statements of the crucial lemmas from [12, 26], as Lemmas 3, 4, and 5 below.

To get our results we need to assume that the pair potential is positive and of finite range R . The positivity in particular allows us to say exactly how small z need be for our results to hold. Recall that we have assumed that the pair potential ϕ satisfies $C := \int [1 - \exp(-\phi(x))] dx < \infty$. From now on, we assume that

$$z < (3eC)^{-1}. \quad (7)$$

This bound is more than sufficient, by [21, Theorems 5.7 and 5.8], to guarantee that μ is the unique Gibbs measure for ϕ and z , and is translation invariant. Now from (5) and (6) we get

$$\sigma_n^2 = \rho n + \int_{V_n} \int_{V_n} \omega_2(x, y) dx dy, \quad (8)$$

where $\omega_2 = \rho_2 - \rho^2$. For z satisfying (7), we have $\omega_2(x, y) = \omega_2(0, y - x)$.

Also, from (5.18) and (5.19) of [20, Section 4.5], we see that $z/(1+Cz) \leq \rho \leq z$, while (4.37) of [20, Section 4.4] (with $B = 0$) gives

$$\int |\omega_2(0, x)| dx \leq z^2 eC / (1 - zeC)^2.$$

Combining these inequalities we see that $\int |\omega_2(0, x)| dx < \rho$ if z satisfies (7). From (8) we conclude that there exist constants $a, b > 0$ so that

$$0 < a \leq \frac{\sigma_n^2}{n} \leq b < \infty. \quad (9)$$

In [24, Lemma 4], an exponential L^2 -mixing for Gibbs fields at low density was proved. We've simplified the expression found there by adjusting the constants c and α . For any two bounded Borel subsets Λ_1 and Λ_2 of $\mathbb{R}^d$ we define $d(\Lambda_1, \Lambda_2) := \inf\{\|x - y\| : x \in \Lambda_1, y \in \Lambda_2\}$. We note that the bound (7) also suffices to get Spohn's result.

Lemma 3 ([24, Lemma 4]) *Let Λ_i be bounded regions ($i = 1, 2$) and suppose $\Psi_i \in \mathcal{F}(\Lambda_i)$ are square integrable. There exist constants $\alpha, c > 0$, depending only on z, ϕ , such that*

$$|\text{Corr}_\mu(\Psi_1, \Psi_2)| \leq \min\{1, c |\Lambda_1^R| e^{-\alpha d(\Lambda_1, \Lambda_2)}\},$$

where $\Lambda^R := \{x \in \mathbb{R}^d : d(x, \Lambda) \leq R\}$.

Lemma 4 *Let $(r_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{Z}^d$ so that $r_i \neq r_j$ if $i \neq j$. Suppose $(\eta_j)_{j \in \mathbb{N}}$ are mean zero, square integrable random variables so that $\eta_j \in \mathcal{F}(Q_{r_j})$ for all $j \in \mathbb{N}$. Suppose also that $\sup_j \text{Var}_\mu(\eta_j) < \infty$. Then there is a constant K so that for any n ,*

$$\text{Var}_\mu \left(\sum_{j=1}^n \eta_j \right) \leq Kn,$$

and any $b \geq 0$,

$$E_\mu \left(\max_{1 \leq j \leq n} \left(\sum_{i=b+1}^{b+j} \eta_i \right)^2 \right) \leq K n (\log 2n)^2.$$

Lemma 5 *Let $\Phi(x) = P(Z > x)$ be the standard normal error function. Then*

$$\sup_{-\infty < x < \infty} |\mu(\sigma_n^{-1} S_n \geq x) - \Phi(x)| = O \left(\frac{1}{(\log n)^3} \right). \quad (10)$$

Also, for any $b > 1$,

$$\mu \left(\max_{1 \leq j \leq n} |S_j| \geq b \chi_n \right) = O \left(\frac{1}{(\log n)^3} \right).$$

Proposition 4 *The law of the iterated logarithm holds for almost every γ , that is, $\mu(LIL^c) = 0$.*

Proof. The assertion will be proved if we show that for any $\epsilon > 0$,

$$\mu(|S_n| > (1 + \epsilon)\chi_n \text{ i.o.}) = 0 \quad (11)$$

and

$$\mu(S_n > (1 - \epsilon)\chi_n \text{ i.o.}) = 1. \quad (12)$$

$$\mu(S_n < -(1 - \epsilon)\chi_n \text{ i.o.}) = 1. \quad (13)$$

The proof of (11) is almost identical to [12, Theorem 1] and therefore is omitted.

We proceed to prove (12). For $k \in \mathbb{N}$ with $k \geq \sigma_1$, define m_k to be the largest integer so $\sigma_{m_k} \leq k^{2k-1}$, and n_k the largest integer so $\sigma_{n_k} \leq k^{2k}$. For $k \geq b+1$, we have $m_k \geq b$ and so using (9)

$$am_k \leq k^{4k-2} \leq b(m_k + 1) \leq (b+1)m_k \quad (14)$$

$$an_k \leq k^{4k} \leq b(n_k + 1) \leq (b+1)n_k. \quad (15)$$

Since $ak^{4k}/(b+1) \leq an_k \leq \sigma_{n_k}^2 \leq k^{4k}$ we have

$$4k \log(k) + \log\left(\frac{a}{b+1}\right) \leq \log(\sigma_{n_k}^2) \leq 4k \log(k). \quad (16)$$

For any $\lambda > 0$ put $B_k = B_k(\lambda) = \{S_{n_k} - S_{m_k} \geq (1-2\lambda)\chi_{n_k}\}$. We want to show that $\sum_k \mu(B_k) = \infty$. We will use the inequality

$$\mu(S_{n_k} \geq (1-\lambda)\chi_{n_k}) \leq \mu(B_k) + \mu(S_{m_k} \geq \lambda\chi_{n_k}). \quad (17)$$

Using $\sigma_{m_k}^2 \leq k^{4k-2}$ and recalling that $\chi_{n_k}^2 \geq \sigma_{n_k}^2 \geq (a/(b+1))k^{4k}$, Chebyshev's inequality gives us

$$\mu(S_{m_k} \geq \lambda\chi_{n_k}) \leq \frac{\sigma_{m_k}^2}{\lambda^2 \chi_{n_k}^2} \leq \frac{b+1}{a\lambda^2 k^2}. \quad (18)$$

Since this is summable it suffices to show that $\sum_k \mu(S_{n_k} \geq (1-\lambda)\chi_{n_k}) = \infty$. From the Central Limit Theorem (10) and (15) we have

$$\sum_k |\mu(S_{n_k} \geq (1-\lambda)\chi_{n_k}) - \Phi((1-\lambda)\chi_{n_k}/\sigma_{n_k})| \leq c \sum_k (\log(n_k))^{-3} < \infty.$$

Therefore it suffices to show that

$$\sum_k \Phi((1-\lambda)\chi_{n_k}/\sigma_{n_k}) = \sum_k \Phi\left((1-\lambda)\sqrt{2 \log \log(\sigma_{n_k}^2)}\right) = \infty.$$

But this follows in the usual way from the asymptotic relation $\Phi(x) \sim \exp(-x^2/2)/x$ and (16).

Let ζ_k be the indicator function of B_k . By virtue of the exponential mixing condition in Lemma 3, and the growth rates (14) and (15) of m_k and n_k , you can show that there is $c > 0$ and $k_0 \in \mathbb{N}$ so that

$$\text{Cov}_\mu(\zeta_k, \zeta_{k+j}) \leq k^{-2} \exp(-cj), \quad \text{for all } k \geq k_0 \text{ and } j \geq 1.$$

Therefore

$$\begin{aligned}
\text{Var}_\mu\left(\sum_{k=k_0}^n \zeta_k\right) &= \sum_{k=k_0}^n \text{Var}_\mu(\zeta_k) + 2 \sum_{k=k_0}^{n-1} \sum_{j=1}^{n-k} \text{Cov}(\zeta_k, \zeta_{j+k}) \\
&\leq \sum_{k=k_0}^n \text{Var}_\mu(\zeta_k) + 2 \sum_{k=1}^{\infty} k^{-2} \sum_{j=1}^{\infty} \exp(-cj) \\
&\leq \sum_{k=k_0}^n \mu(B_k) + K.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu\left(\sum_{k=k_0}^{\infty} \zeta_k \leq \frac{1}{2} \sum_{k=k_0}^n \mu(B_k)\right) &\leq \mu\left(\sum_{k=k_0}^n \zeta_k \leq \frac{1}{2} \sum_{k=k_0}^n \mu(B_k)\right) \\
&\leq \mu\left(\left|\sum_{k=k_0}^n \zeta_k - \sum_{k=k_0}^n \mu(B_k)\right| \geq \frac{1}{2} \sum_{k=k_0}^n \mu(B_k)\right) \\
&\leq \frac{4\text{Var}_\mu(\sum_{k=k_0}^n \zeta_k)}{(\sum_{k=k_0}^n \mu(B_k))^2} \\
&\leq \frac{4(\sum_{k=k_0}^n \mu(B_k) + K)}{(\sum_{k=k_0}^n \mu(B_k))^2}.
\end{aligned}$$

Since $\sum_k \mu(B_k) = \infty$, letting $n \rightarrow \infty$ gives $\mu(\sum_{k=k_0}^{\infty} \zeta_k < \infty) = 0$ so

$$\mu(B_k(\lambda) \text{ i.o.}) = 1. \quad (19)$$

Note that $B_k(\epsilon/4) \subset (S_{n_k} \geq (1-\epsilon)\chi_{n_k}) \cup (-S_{m_k} \geq (\epsilon/2)\chi_{n_k})$, so from (19)

$$1 \leq \mu(S_{n_k} \geq (1-\epsilon)\chi_{n_k} \text{ i.o.}) + \mu(-S_{m_k} \geq (\epsilon/2)\chi_{n_k} \text{ i.o.}).$$

But as in (18) we see that $\sum_k \mu(-S_{m_k} \geq (\epsilon/2)\chi_{n_k}) < \infty$ so that $\mu(-S_{m_k} \geq (\epsilon/2)\chi_{n_k} \text{ i.o.}) = 0$. From this (12) follows and (13) can be proved similarly. Therefore the theorem is proved.

Proposition 5 *In dimensions $d \leq 3$, the set LIL^c is $\mathcal{E}$ -exceptional.*

Proof. Let $n_1 = 1$ and for $k \geq 1$ let $n_{k+1} = [n_k + \sqrt{n_k}]$. For $n > 1$, choose k so that $n_k \leq n \leq n_{k+1}$ and hence

$$\left(\frac{\chi_{n_k}}{\chi_{n_{k+1}}}\right)\left(\frac{S_{n_k}}{\chi_{n_k}}\right) + \left(\frac{\mu_{n_k} - \mu_{n_{k+1}}}{\chi_{n_{k+1}}}\right) < \frac{S_n}{\chi_n} \leq \left(\frac{\chi_{n_{k+1}}}{\chi_{n_k}}\right)\left(\frac{S_{n_{k+1}}}{\chi_{n_{k+1}}}\right) + \left(\frac{\mu_{n_{k+1}} - \mu_{n_k}}{\chi_{n_k}}\right).$$

Using the formula $\mu_{n_k} = \rho n_k$ and $\chi_{n_k} = \sqrt{2n_k \log \log(n_k)}$, the inequality above shows that $\limsup_n (S_n/\chi_n) = \limsup_k (S_{n_k}/\chi_{n_k})$.

We now construct a smooth approximation for S_{n_k}/χ_{n_k} . First we bound below the distance between the cubes $V_{n_{k+1}}$ and V_{n_k} :

$$n_{k+1}^{1/d}/2 - n_k^{1/d}/2 \geq \frac{n_k^{1/d-1/2}}{16d}.$$

So for every $k \geq 1$, we can choose a smooth function ψ_k satisfying

$$1_{[-n_k^{1/d}/2, n_k^{1/d}/2]} \leq \psi_k \leq 1_{[-n_{k+1}^{1/d}/2, n_{k+1}^{1/d}/2]},$$

and $|\psi'_k| \leq 17dn_k^{1/2-1/d}$. Define a continuous element of $D(\mathcal{E})$ by

$$u_k(\gamma) := (\langle \prod_{i=1}^d \psi_k(x_i), \gamma \rangle - \mu_{n_k})/\chi_{n_k}.$$

Taking the limsup through the inequality

$$\left(\frac{S_{n_k}}{\chi_{n_k}} \right) \leq u_k \leq \left(\frac{\chi_{n_{k+1}}}{\chi_{n_k}} \right) \left(\frac{S_{n_{k+1}}}{\chi_{n_{k+1}}} \right) + \left(\frac{\mu_{n_{k+1}} - \mu_{n_k}}{\chi_{n_k}} \right).$$

we find that $u(\gamma) := \limsup_k u_k(\gamma) = \limsup_k S_{n_k}(\gamma)/\chi_{n_k}$.

Bounding the square field gives

$$\boxplus(u_k)(\gamma) \leq \left(\frac{17d^2 n_k^{1/2-1/d}}{\chi_{n_k}} \right)^2 \gamma(V_{n_{k+1}} \setminus V_{n_k}) \leq c \frac{\gamma(V_{n_{k+1}} \setminus V_{n_k})}{n_k^{2/d}}. \quad (20)$$

Using the definition of n_k and since $|V_n| = n$, we get

$$\begin{aligned} \frac{\gamma(V_{n_{k+1}} \setminus V_{n_k})}{n_k^{2/d}} &\leq \frac{\gamma(V_{n_{k+1}} \setminus V_{n_k}) - \rho|V_{n_{k+1}} \setminus V_{n_k}|}{n_k^{2/d}} + \frac{\rho|V_{n_{k+1}} \setminus V_{n_k}|}{n_k^{2/d}} \\ &\leq \left(\frac{n_{k+1}}{n_k} \right)^{2/d} \frac{|\gamma(V_{n_{k+1}}) - \rho|V_{n_{k+1}}||}{n_{k+1}^{2/d}} + \frac{|\gamma(V_{n_k}) - \rho|V_{n_k}||}{n_k^{2/d}} + \rho \frac{n_k^{1/2}}{n_k^{2/d}} \\ &\leq 4 \frac{|\gamma(V_{n_{k+1}}) - \rho|V_{n_{k+1}}||}{n_{k+1}^{2/d}} + \frac{|\gamma(V_{n_k}) - \rho|V_{n_k}||}{n_k^{2/d}} + \rho \frac{n_k^{1/2}}{n_k^{2/d}}. \end{aligned} \quad (21)$$

We need some notation so that we can work with the more convenient cubes C_n . Let k_0 be so large that $n_{k_0} \geq 4^d$ and for $k \geq k_0$ let m_k be the integer that satisfies

$$[2(m_k - 1)]^d < n_k \leq (2m_k)^d.$$

Then $C_{m_k-1} \subset V_{n_k} \subset C_{m_k}$. The condition on k_0 guarantees that $m_k \geq 2$, then we get

$$\begin{aligned}
\frac{\gamma(V_{n_k}) - \rho|V_{n_k}|}{n_k^{2/d}} &\leq \left(\frac{|C_{m_k}|}{n_k} \right)^{2/d} \frac{|\gamma(C_{m_k}) - \rho|C_{m_k}||}{|C_{m_k}|^{2/d}} + \frac{\rho||C_{m_k}| - |V_{n_k}||}{n_k^{2/d}} \\
&\leq 4 \frac{|\gamma(C_{m_k}) - \rho|C_{m_k}||}{|C_{m_k}|^{2/d}} + \frac{\rho 2^d [m_k^d - (m_k - 1)^d]}{4(m_k - 1)^2} \\
&\leq 4 \frac{|\gamma(C_{m_k}) - \rho|C_{m_k}||}{|C_{m_k}|^{2/d}} + c m_k^{d-3}. \tag{22}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\rho|V_{n_k}| - \gamma(V_{n_k})}{n_k^{2/d}} &\leq \left(\frac{|C_{m_k-1}|}{n_k} \right)^{2/d} \frac{|\gamma(C_{m_k-1}) - \rho|C_{m_k-1}||}{|C_{m_k-1}|^{2/d}} + \frac{\rho||C_{m_k-1}| - |V_{n_k}||}{n_k^{2/d}} \\
&\leq \frac{|\gamma(C_{m_k-1}) - \rho|C_{m_k-1}||}{|C_{m_k-1}|^{2/d}} + c m_k^{d-3}. \tag{23}
\end{aligned}$$

Combined with (20), (21), (22), and (23) this gives us, for $d \leq 3$,

$$\sup_{k \geq k_0} \left(\boxplus(u_k)(\gamma) \right) \leq c \left(1 + \sup_{k \geq k_0} \frac{|\gamma(C_{m_k}) - \rho|C_{m_k}||}{|C_{m_k}|^{2/d}} \right). \tag{24}$$

As in the proof of Proposition 2, let $(r_m)_{m \in \mathbb{N}}$ be a sequence in $\mathbb{Z}^d$ so $C_n = \cup_{m=1}^{|C_n|} Q_{r_m}$ for $n \in \mathbb{N}$. Denote $\eta_m(\gamma) = \gamma(Q_{r_m}) - \rho$ for $m \in \mathbb{N}$. By applying Lemma 4, for $d \leq 3$, we have

$$\begin{aligned}
\int_{\tilde{\Gamma}_{\mathbb{R}^d}} \sup_{n \in \mathbb{N}} \frac{|\gamma(C_n) - \rho|C_n||}{|C_n|^{2/d}} \mu(d\gamma) &= \int_{\tilde{\Gamma}_{\mathbb{R}^d}} \sup_{n \in \mathbb{N}} \frac{|\eta_1 + \cdots + \eta_{|C_n|}|}{|C_n|^{2/d}} \mu(d\gamma) \\
&\leq \int_{\tilde{\Gamma}_{\mathbb{R}^d}} \sup_{m \in \mathbb{N}} \frac{|\eta_1 + \cdots + \eta_m|}{m^{2/3}} \mu(d\gamma) \\
&= \lim_{k \rightarrow \infty} \int_{\tilde{\Gamma}_{\mathbb{R}^d}} \max_{m \leq 2^k} \frac{|\eta_1 + \cdots + \eta_m|}{m^{2/3}} \mu(d\gamma) \\
&\leq \lim_{k \rightarrow \infty} \sum_{l=1}^k \int_{\tilde{\Gamma}_{\mathbb{R}^d}} \max_{2^{l-1} < m \leq 2^l} \frac{|\eta_1 + \cdots + \eta_m|}{m^{2/3}} \mu(d\gamma) \\
&\leq \sum_{l=1}^{\infty} \frac{1}{2^{2(l-1)/3}} \int_{\tilde{\Gamma}_{\mathbb{R}^d}} \max_{2^{l-1} < m \leq 2^l} |\eta_1 + \cdots + \eta_m| \mu(d\gamma)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{l=1}^{\infty} \frac{1}{2^{2(l-1)/3}} E_{\mu} \left(\max_{m \leq 2^l} (\eta_1 + \dots + \eta_m)^2 \right)^{1/2} \\
&\leq \sum_{l=1}^{\infty} \frac{1}{2^{2(l-1)/3}} (c 2^l \log(2^{l+1}))^{1/2} \\
&\leq c \sum_{l=1}^{\infty} (l+1) 2^{-l/6} < \infty.
\end{aligned}$$

Combining this estimate with Proposition 4, the proof is almost identical with that of Proposition 2 and therefore is omitted.

Remark. Röckner and Schied [17] have also recently proved a capacitary version of the law of large numbers on configuration space using an interesting approach based on an intrinsic metric ρ . Indeed, it is easy to see that $LLN^c \cap \Gamma_{\mathbb{R}^d} = \{\gamma \in \Gamma_{\mathbb{R}^d} : \rho(\gamma, \omega) < \infty \text{ for some } \omega \in LLN^c \cap \Gamma_{\mathbb{R}^d}\}$ so by [17, Corollary 3.2], $\mu(LLN^c) = 0$ implies that $LLN^c \cap \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$ -exceptional. Their technique will also give a capacitary version of the law of the iterated logarithm.

However, the Dirichlet form used in [17] extends $(\mathcal{E}, D(\mathcal{E}))$, but it is not known whether the forms coincide. A form with a larger domain has more exceptional sets, so proofs of exceptionality are easier in Röckner and Schied's setting, and do not imply exceptionality in our setting.

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Byron Schmuland
 Department of Mathematical Sciences
 University of Alberta
 Edmonton, Alberta, Canada T6G 2G1
 schmu@stat.ualberta.ca

Wei Sun
 Institute of Applied Mathematics
 Chinese Academy of Sciences
 Beijing, China 100080
 and

Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta, Canada T6G 2G1
`wsun@stat.ualberta.ca`