On the equation $\mu_{t+s} = \mu_s * T_s\mu_t$

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Abstract

We prove that any solution of the equation $\mu_{t+s} = \mu_s * T_s\mu_t$ consists of infinitely divisible measures, and that there are constants $(b_t)_{t\geq 0}$ so that $t \mapsto \mu_t * \delta_{-b_t}$ is continuous.

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1 Introduction

A convolution semigroup is a family $(\mu_t)_{t\geq 0}$ of probability measures satisfying $\mu_{t+s} = \mu_s * \mu_t$ for $s, t \geq 0$. The measures μ_t are clearly infinitely divisible, however $t \mapsto \mu_t$ need not be continuous. The classic counterexample is $\mu_t = \delta_{b_t}$ where $t \mapsto b_t$ is a discontinuous solution of the functional equation $b_{t+s} = b_t + b_s$. Fortunately, for convolution semigroups this is the extent of the pathology as $t \mapsto \mu_t$ can always be made continuous by removal of a non-random solution of $b_{t+s} = b_t + b_s$ (Breiman, 1992, Section 14.4).

A (T_t)-convolution semigroup is a family of probability measures satisfying the more general equation (1). The goal of this paper is to extend the results on infinite divisibility and continuity from convolution semigroups to (T_t)-convolution semigroups.

Let E be a real separable Banach space, and let E^* denote its dual. By $\mathcal{B}(E)$ we mean the Borel subsets on E and by $\mathcal{B}_b(E)$ the set of bounded Borel measurable functions from E into $\mathbb{R}$. We let $\widehat{\mu}$ denote the characteristic function of the probability measure μ on $\mathcal{B}(E)$, and $\mu * \nu$ denote the convolution of two probability measures. We say μ is a factor of ν, and write $\mu \prec \nu$, if $\nu = \mu * \sigma$ for some probability measure σ. For each probability measure μ we let $\mu^-(B) := \mu(-B)$, and define a symmetrization of μ by $\widetilde{\mu} = \mu * \mu^-$.

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of linear operators on E and $(\mu_t)_{t\geq 0}$ a family of probability measures on E. Using the notation $T_s\mu_t$ for the induced measure $\mu_t \circ T_s^{-1}$, we say that $(\mu_t)_{t\geq 0}$ is a (T_t)-*convolution semigroup* (or more precisely, a (T_t)-skew convolution semigroup) if it satisfies

$$\mu_{t+s} = \mu_s * T_s\mu_t, \quad s, t \geq 0. \tag{1}$$

One motivation for the equation (1) follows from the fact that any solution leads to a Markov semigroup $(p_t)_{t\geq 0}$ of operators, called a *(generalized) Mehler semigroup*, given by

$$p_t f(x) = \int_E f(T_t x + y)\, \mu_t(dy), \quad x \in E, f \in \mathcal{B}_b(E).$$

This generalization of the Ornstein-Uhlenbeck semigroup on Banach space was studied by Bogachev and Röckner (1995) and Bogachev et al. (1996). In the most general work to date, Fuhrman and Röckner (1997) assume the measures μ_t to be infinitely divisible, and the map $t \mapsto \widehat{\mu}_t(a)$ to be continuously differentiable.

There is also an important body of work that looks at equation (1) for measures μ_t on a space of measures, in the context of measure-valued branching processes. In this setting Z.H. Li (1996) proved that $t \mapsto \widehat{\mu}_t(a)$ is continuous and he gave a representation of $\widehat{\mu}_t(a)$ in terms of an infinitely divisible entrance law. The connection between these measure-valued processes and Ornstein-Uhlenbeck type processes has also been studied by Gorostiza and Li (1998, 2000).

2 Infinite divisibility

Proposition 1. *If $(\mu_t)_{t\geq 0}$ is a (T_t)-convolution semigroup, then the measures $(\mu_t)_{t\geq 0}$ are infinitely divisible.*

Proof. It suffices to show that μ_1 is infinitely divisible. For each $n \in \mathbb{N}$ and probability measure μ on E define the convolution product $F_n(\mu) = \prod_{j=0}^{2^n-1} T_{j/2^n}\mu$. Note that $\mu \prec F_n(\mu)$ and that $F_{n+1}(\mu) = F_n(\mu) * F_n(T_{1/2^{n+1}}\mu)$ so by induction $F_n(\mu) \prec F_m(\mu)$ for $m \geq n$. We also have $F_n(\mu * \delta_x) = F_n(\mu) * \delta_{\Sigma_j T_{j/2^n} x}$. Since $(\mu_t)_{t\geq 0}$ is a (T_t)-convolution semigroup we have $\mu_1 = F_m(\mu_{1/2^m})$ for all $m \in \mathbb{N}$. Then we get $\mu_{1/2^m} \prec F_m(\mu_{1/2^m}) = \mu_1$ so from Linde (1986, Proposition 2.4.1) the sequence $\mu_{1/2^m}$ is relatively shift compact. Let $m_k \to \infty$, $x_k \in E$, and ν a probability on E such that $\mu_{1/2^{m_k}} * \delta_{x_k} \Rightarrow \nu$. For $m_k \geq n$,

$$|\widehat{\mu}_1(a)| = |\widehat{F_{m_k}}(\mu_{1/2^{m_k}})(a)| \leq |\widehat{F_n}(\mu_{1/2^{m_k}})(a)| = |\widehat{F_n}(\mu_{1/2^{m_k}} * \delta_{x_k})(a)|.$$

Taking the limit as $k \to \infty$ gives $|\widehat{\mu}_1(a)| \leq |\widehat{F_n}(\nu)(a)|$. In particular, for small enough $\varepsilon > 0$, $\|a\|_{E^*} \leq \varepsilon$ implies $1/2 \leq |\widehat{\mu}_1(a)| \leq |\widehat{F_n}(\nu)(a)|$. For $k \in \mathbb{N}$ define $z_k = \prod_j \{|\widehat{\nu}(T^*_{j/2^k}a)| : 1 \leq j \leq 2^k, j \text{ odd}\}$. Then $0 \leq z_k \leq 1$, and the sequence $|\widehat{F}_n(\nu)(a)| = |\widehat{\nu}(a)| \prod_{k=1}^n z_k$ is decreasing and, by hypothesis, bounded away from zero. This means that $z_k \to 1$, so letting $k \to \infty$ in $z_k \leq |\widehat{\nu}(T^*_{1/2^k}a)| \leq 1$ we obtain $|\widehat{\nu}(a)| = 1$.

This last step requires a word of explanation. Since the dual semigroup $(T^*_t)_{t\geq 0}$ need not be strongly continuous, we have no guarantee that $T^*_{1/2^k}a \to a$ in E^* norm. However these functionals certainly converge pointwise as $\langle T^*_{1/2^k}a, x\rangle = \langle a, T_{1/2^k}x\rangle \to \langle a, x\rangle$ as $k \to \infty$. Also, since $\sup_k \|T^*_{1/2^k}\|_{\mathcal{L}(E^*)} = \sup_k \|T_{1/2^k}\|_{\mathcal{L}(E)} < \infty$, the functionals $T^*_{1/2^k}a$ are equicontinuous, so by Ascoli's theorem they converge uniformly on compacts. But this is all we need, since the characteristic function $\widehat{\nu}$ is continuous when E^* is equipped with this topology (Linde (1986, Proposition 1.7.2)).

The symmetrized measure satisfies $\widehat{\widetilde{\nu}}(a) = |\widehat{\nu}(a)|^2 = 1$ for $\|a\|_{E^*} \leq \varepsilon$. Then Proposition 1.7.6 of Linde (1986) implies that $\widetilde{\nu} = \delta_0$ and applying Linde (1986, Proposition

2.1.3) shows that ν itself is degenerate, say $\nu = \delta_x$. By subtracting x from the sequence x_k, we may assume that $\mu_{1/2^{m_k}} * \delta_{x_k} \Rightarrow \delta_0$.

Combined with the fact that $\sup\{\|T_t\|_{\mathcal{L}(E)} : 0 \le t \le 1\} < \infty$ this shows that the measures $\nu_{kj} := T_{j/2^{m_k}}(\mu_{1/2^{m_k}} * \delta_{x_k})$, $(k \in \mathbb{N},\ 0 \le j \le 2^{m_k} - 1)$ form an infinitesimal triangular array. In addition, $\prod_j \nu_{kj} = F_{m_k}(\mu_{1/2^{m_k}} * \delta_{x_k}) = \mu_1 * \delta_{\Sigma_j T_{j/2^{m_k}}(x_k)}$, in other words $\prod_j \nu_{kj} * \delta_{-\Sigma_j T_{j/2^{m_k}}(x_k)} = \mu_1$ and hence the result follows from de Acosta et al. (1978, Theorem 2.10). □

By the Lévy-Khinchine theorem (Linde (1986, Theorem 5.7.3)), the characteristic function of an infinitely divisible measure μ can be expressed as $\widehat{\mu}(a) = \exp(-\psi(a))$ where ψ is called the *characteristic exponent* of μ, and

$$\psi(a) = -i\langle a, b\rangle + \frac{1}{2}\langle a, Ra\rangle - \int_{E\setminus\{0\}} K(a,x)\, M(dx). \tag{2}$$

Here $b \in E$, $R : E^* \to E$ is the covariance of a Gaussian measure on E, and M is a Lévy measure on $E \setminus \{0\}$. The integrand is

$$K(a,x) := e^{i\langle a,x\rangle} - 1 - i\langle a, x\rangle \chi_{[0,1]}(\|x\|).$$

The triple (b, R, M) is uniquely determined by μ. The expression in (2) shows that μ can be decomposed into the convolution of three infinitely divisible measures $\mu = \delta_b * g * j$, where

$$\psi_{\delta_b}(a) = -i\langle a, b\rangle, \quad \psi_g(a) = \frac{1}{2}\langle a, Ra\rangle, \quad \psi_j(a) = -\int_{E\setminus\{0\}} K(a,x)\, M(dx).$$

We call δ_b the constant part, g the Gaussian part, and j the jump part of the measure μ. Returning to our equation, from (2) it is easy to show that $(\mu_t)_{t\ge0}$ solves (1) if and only if,

$$\psi_{t+s}(a) = \psi_s(a) + \psi_t(T_s^* a). \tag{3}$$

By uniqueness of the decomposition, this gives

$$\begin{aligned} b_{t+s} &= b_s + T_s b_t + \int_{E\setminus\{0\}} (T_s x)[\chi_{[0,1]}(\|T_s x\|) - \chi_{[0,1]}(\|x\|)]\, M_t(dx), && (4)\\ R_{t+s} &= R_s + T_s R_t T_s^*, && (5)\\ M_{t+s} &= M_s + (T_s M_t)|_{E\setminus\{0\}}. && (6) \end{aligned}$$

If $(\mu_t)_{t\ge0}$ solves (1), then (5) and (3) show that the Gaussian part $(g_t)_{t\ge0}$ also solves (1). Unfortunately, equation (4) shows that b_t and M_t are more intimately tied together, and that in general the constant part $(\delta_{b_t})_{t\ge0}$ and the jump part $(j_t)_{t\ge0}$ need not solve equation (1) separately.

3 Continuity

Throughout this section we fix a (T_t)-convolution semigroup $(\mu_t)_{t\ge0}$ with decomposition $\mu_t = \delta_{b_t} * g_t * j_t$. All integrations take place over $E \setminus \{0\}$.

3.1 Continuity of $t \mapsto \psi_{g_t}(a)$

Lemma 1. *Suppose $f : [0,\infty) \to \mathbb{R}$ satisfies*

$$f(t+s) = f(s) + c(t,s), \tag{7}$$

where c is a non-negative function that is continuous in s for every fixed t. Then f is continuous.

Proof. Since c is non-negative, (7) tells us that f is non-decreasing. Fix $t \geq 0$, $s > 0$, and let $\varepsilon < s$. Equation (7) gives us $f(t+(s+\varepsilon)) = f(s+\varepsilon) + c(t, s+\varepsilon)$ and $f(t+(s-\varepsilon)) = f(s-\varepsilon) + c(t, s-\varepsilon)$. Taking the difference gives

$$f(t+s+\varepsilon) - f(t+s-\varepsilon) = [f(s+\varepsilon) - f(s-\varepsilon)] + [c(t,s+\varepsilon) - c(t,s-\varepsilon)],$$

and letting $\varepsilon \to 0$ we obtain $(\triangle f)(t+s) = (\triangle f)(s)$. Since the jump size $\triangle f$ must be zero at all but countably many points, we conclude that $(\triangle f)(s) = 0$. A similar argument shows that $(\triangle f)(0) = 0$, and we conclude that f is continuous. □

Lemma 2. *If $(\mu_t)_{t\geq 0}$ is a (T_t)-convolution semigroup, then $t \mapsto |\widehat{\mu}_t(a)|$ is continuous for every $a \in E^*$. In particular, if μ_t are symmetric, then $t \mapsto \widehat{\mu}_t(a)$ is continuous.*

Proof. Since μ_t is infinitely divisible, we have $|\widehat{\mu}_t(a)| = \exp(-\operatorname{Re}\psi_t(a))$, where $\operatorname{Re}\psi_t(a)$ is non-negative. Applying Lemma 1 to the real part of (2) yields the result. As in Proposition 1, we use the fact that $s \mapsto \psi_t(T_s^* a)$ is continuous, even though the dual semigroup $(T_s^*)_{s\geq 0}$ may not be continuous.

The last statement follows since $|\widehat{\mu}_t(a)| = \widehat{\mu}_t(a)$ for symmetric infinitely divisible measures. □

Proposition 2. *The map $t \mapsto \psi_{g_t}(a)$ is continuous for every $a \in E^*$.*

Proof. This follows since the Gaussian part $(g_t)_{t\geq 0}$ is a symmetric solution of (1). □

3.2 Continuity of $t \mapsto \psi_{j_t}(a)$

The proof that $t \mapsto \psi_{j_t}(a)$ is continuous uses similar ideas but is more complicated. The essential condition is found in equation (6) which places a powerful restriction on the family of measures $(M_t)_{t\geq 0}$. On the other hand, we must proceed with caution, as we need to apply dominated convergence with the (possibly) infinite measures M_t. If E were a Hilbert space, then $\int (\|x\|^2 \wedge 1)\, M_t(dx) < \infty$ for every t (Linde, 1986, Remark page 74), and the proof could be shortened.

Lemma 3. *Suppose that $g : [0,\infty) \times E \to \mathbb{R}$ is continuous in s for each $x \in E$ and that for every n there are constants $c_n, d_n > 0$ so that $\sup_{0\leq s\leq n} |g(s,x)| \leq c_n\, \chi_{[d_n,\infty)}(\|x\|)$. Then the function $c(s,t) := \int g(s,x)\, M_t(dx)$ is continuous in s.*

Proof. This is a simple application of the dominated convergence theorem. □

Lemma 4. *Suppose that g is a bounded function with compact support in $E \setminus \{0\}$. Then $f(t) := \int g(x)\, M_t(dx)$ is continuous.*

Proof. We first consider the case where g is non-negative and continuous. Equation (6) gives us $f(t+s) = f(s) + c(s,t)$ where $c(s,t) = \int g(T_s x)\, M_t(dx)$. Since g vanishes near the origin and since $\sup_{0\le s\le n} \|T_s\|_{\mathcal{L}(E)} < \infty$ the bound in Lemma 3 holds, so that $s \mapsto c(s,t)$ is continuous. The continuity of f follows from Lemma 1, in particular, $\int g(x)\, M_\varepsilon(dx) \to 0$ as $\varepsilon \to 0$.

For general g we have

$$f(t+\varepsilon) = f(t) + \int g(T_t x)\, M_\varepsilon(dx), \qquad f(t) = f(t-\varepsilon) + \int g(T_{t-\varepsilon} x)\, M_\varepsilon(dx).$$

Let h be a continuous function with compact support in $(0,\infty)$ so that

$$\sup_{0\le t\le n} |g(T_t x)| \le h(\|x\|).$$

Then for $0 \le t < t+\varepsilon \le n$ we have $|f(t+\varepsilon) - f(t)| \le \int h(\|x\|)\, M_\varepsilon(dx)$, which goes to zero as $\varepsilon \to 0$, by the first part of the proof. This shows that f is right continuous.

Similarly, for $0 \le t-\varepsilon < t \le n$, we have $|f(t) - f(t-\varepsilon)| \le \int h(\|x\|)\, M_\varepsilon(dx) \to 0$, as $\varepsilon \to 0$. This shows that f is left continuous. $\square$

Next we introduce a continuous version of the characteristic exponent ψ. Let ϕ be a continuous function on $\mathbb{R}$ so that $\chi_{[0,1]} \le \phi \le \chi_{[0,2]}$ and define

$$\begin{aligned} K'(a,x) &= e^{i\langle a,x\rangle} - 1 - i\langle a,x\rangle \phi(\|x\|), \\ \psi'_{j_t}(a) &= \int K'(a,x)\, M_t(dx). \end{aligned}$$

Remark 1. Note that $\psi'_{j_t}(a) = \psi_{j_t}(a) - i\langle a, d_t\rangle$ where $d_t = \int x[\phi - \chi_{[0,1]}](\|x\|)\, M_t(dx)$. Applying Lemma 4 to $g(x) := \langle a,x\rangle(\phi - \chi_{[0,1]})(\|x\|)$ shows that $t \mapsto \langle a, d_t\rangle$ is continuous. In other words, to prove the continuity of $t \mapsto \psi_{j_t}(a)$, it suffices to show that $t \mapsto \psi'_{j_t}(a)$ is continuous.

Lemma 5. *For fixed $a \in E^*$ and $t \ge 0$, the map $s \mapsto \int \operatorname{Im} K'(a, T_s x)\, M_t(dx)$ is continuous.*

Proof. We have

$$\int \operatorname{Im} K'(a, T_s x)\, M_t(dx) = \operatorname{Im} \psi'_{j_t}(T_s^* a) + \int \langle a, T_s x\rangle [\phi(\|x\|) - \phi(\|T_s x\|)]\, M_t(dx).$$

Since ψ'_{j_t} is the characteristic exponent of an infinitely divisible measure on E, as in the proof of Proposition 1, $s \mapsto \psi'_{j_t}(T_s^* a)$ is continuous for fixed t and $a \in E^*$. On the other hand, Lemma 3 applies to the integral on the right hand side, giving us the result. $\square$

Proposition 3. *The map $t \mapsto \psi_{j_t}(a)$ is continuous for every $a \in E^*$.*

Proof. Applying Lemma 2 to the (T_t)-convolution semigroup $\delta_{b_t} * j_t$, we find that $t \mapsto \operatorname{Re} \psi'_{j_t}(a) = \operatorname{Re} \psi_{j_t}(a)$ is continuous.

Define $f^+(t) := \int [\operatorname{Im} K'(a,x)]^+\, M_t(dx)$. Equation (6) gives us $f^+(t+s) = f^+(s) + c^+(s,t)$ where

$$c^+(s,t) = \int [\operatorname{Im} K'(a, T_s x)]^+\, M_t(dx) = \int [\sin(\langle a, T_s x\rangle) - \langle a, T_s x\rangle \phi(\|T_s x\|)]^+\, M_t(dx).$$

The integrand is non-negative, continuous in s, and satisfies

$$\sup_{0\le s\le n} |[\operatorname{Im} K'(a, T_s x)]^+| \le (2\|a\|+1)\chi_{[d_n,\infty)}(\|x\|),$$

where $d_n = (\sup\{\|T_s\|_{\mathcal{L}(E)} : 0 \le s \le n\})^{-1}$. Therefore Lemma 3 shows us that $s \mapsto c^+(s,t)$ is continuous and from Lemma 1, $t \mapsto f^+(t)$ is continuous.

This time let $f^-(t) := \int [\operatorname{Im}(K'(a,x))]^- \, M_t(dx)$. Once again (6) gives us $f^-(t+s) = f^-(s) + c^-(s,t)$ where

$$c^-(s,t) = \int [\operatorname{Im} K'(a, T_s x)]^- \, M_t(dx) = c^+(s,t) - \operatorname{Im} \int K'(a, T_s x)\, M_t(dx).$$

The function $c^-(s,t)$ is non-negative and is continuous in s by the previous paragraph and Lemma 5. From Lemma 1, we conclude that $t \mapsto f^-(t)$ is continuous.

Therefore $t \mapsto \psi'_{j_t}(a) = \operatorname{Re} \psi'_{j_t}(a) + i(f^+(t) - f^-(t))$ is continuous, and by Remark 1 we know that $t \mapsto \psi_{j_t}(a)$ is continuous. □

3.3 Continuity of $t \mapsto \psi_{\delta_{b_t}}(a)$

Propositions 2 and 3 tell us that for every (T_t)-convolution semigroup $(\mu_t)_{t\ge 0}$ the map $t \mapsto \psi_t(a) - \psi_{\delta_{b_t}}(a) = \psi_{g_t}(a) + \psi_{j_t}(a)$ is continuous for $a \in E^*$. Therefore the continuity of $\psi_t(a)$ hinges on the continuity of $t \mapsto \psi_{\delta_{b_t}}(a)$, that is, of $t \mapsto b_t$. As explained in the first paragraph of the paper, there can exist discontinuous solutions.

At the present, there is no complete characterization of all discontinuous solutions of equation (1). However, we expect that those described at the start of this paper are essentially the only ones. More concretely, it is not hard to prove that if $I - T_t$ is invertible for some $t > 0$, then there are no discontinuous solutions. The intuition is that discontinuous solutions can exist only insofar as the operators T_t act as the identity.

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