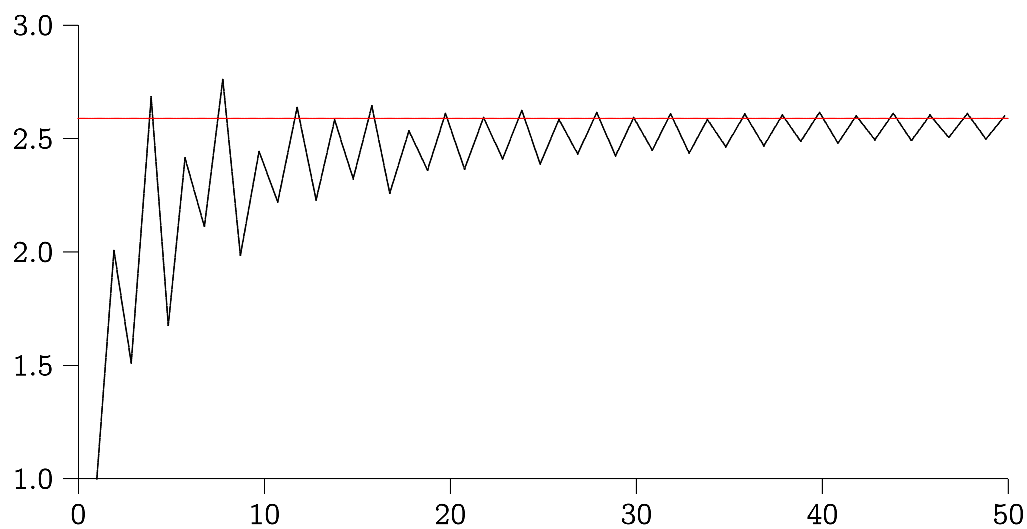
My recent question on `http://math.stackexchange.com/`

Some years ago I was interested in the following Markov chain whose state space is the positive integers. The chain begins at state 1, and from state n the chain next jumps to a state uniformly selected from $\{n+1, n+2, \ldots, 2n\}$. As time goes on, this chain goes to infinity, with occasional large jumps. In any case, the chain is quite unlikely to hit any particular large n.

If you define $p(n)$ to be the probability that this chain visits state n, then $p(n)$ goes to zero like c/n for some constant c. In fact, it seems that

$$np(n) \to c = \frac{1}{2\log(2) - 1} = 2.588699. \tag{1}$$

Here is a graph of this sequence.



To try to prove convergence, I recast it as an analytic problem. Using the Markov property, you can see that the sequence satisfies:

$$p(1) = 1 \quad \text{and} \quad p(n) = \sum_{j=\lceil n/2 \rceil}^{n-1} \frac{p(j)}{j} \text{ for } n > 1. \tag{2}$$

For some weeks, using generating functions etc. I tried and failed to find an analytic proof of the convergence in (1). Finally, at a conference in 2003 Tom Mountford showed me a (non-trivial) probabilistic proof.

So the result is true, but since then I've continued to wonder if I missed something obvious. Perhaps there is a standard technique for showing that (2) implies (1).

Analytic attempt #1 Define Q to be the generating function of $p(n)/n$, that is, $Q(t) = \sum_{n=1}^{\infty} \frac{p(n)}{n} t^n$ for $0 \le t < 1$. Differentiating gives

$$Q'(t) = 1 + \frac{Q(t) - Q(t^2)}{1-t}.$$

Differentiating again and multiplying by $1-t$, we get

$$(1-t)Q''(t) = -1 + 2\left[Q'(t) - tQ'(t^2)\right],$$

that is,

$$(1-t)\sum_{j=0}^{\infty}(j+1)p(j+2)t^j = -1 + 2\left[\sum_{j=1}^{\infty}(jp(j))\frac{t^j - t^{2j}}{j}\right].$$

Assume that $\lim_n np(n) = c$ exists. Letting $t \to 1$ above the left hand side gives c, while the right hand side is $-1 + 2c\log(2)$ and hence $c = \frac{1}{2\log(2)-1}$. Note that $\sum_{j=1}^{\infty} \frac{t^j - t^{2j}}{j} = \log(1+t)$.

Analytic attempt #2 Here's an alternative proof of the conditional result that my colleague Terry Gannon showed me in 2003. Start with the sum $\sum_{n=2}^{2N} p(n)$, substitute the formula in the title, exchange the variables j and n, and rearrange to establish the identity:

$$\frac{1}{2} = \sum_{j=N+1}^{2N} \frac{j-N}{j}\, p(j).$$

If $jp(j) \to c$, then $1/2 = \lim_{N\to\infty} \sum_{j=N+1}^{2N} \frac{j-N}{j^2}\, c = (\log(2) - 1/2)\, c$, so that $c = \frac{1}{2\log(2)-1}$.

Tom Mountford's probabilistic argument: We begin by expressing $xp(x)$ as an expectation involving the process $Z_n := \log(X_n)$,

$$xp(x) = \mathbb{E}\left(\sum_{-\log(2)\le Z_n - \log(x) < 0} \exp(-[Z_n - \log(x)])\right).$$

The convergence follows since Z_n is almost a random walk, for which the renewal theorem (see page 172 of Kallenberg's book) guarantees convergence of such expectations. However, the increments of Z_n are neither independent, nor identically distributed, so we will couple Z_n with a genuine random walk W_n.

The bulk of the proof consists of the details needed to show that the approximation $Z_n \approx W_n$ is close enough.

Preliminary results on occupation measures

1. An increasing random walk W_n.

For any process Y_n we define the occupation measure $\eta^Y = \sum_{n\geq 0} \delta_{Y_n}$. We also define the shifted measure $\theta_x \eta^Y = \sum_{n\geq 0} \delta_{Y_n - x}$, and the measure η_0^Y for η^Y conditioned on $Y_0 = 0$.

For any $\mathbb{R}^d$-valued, discrete time parameter Markov process Y_n, with arbitrary starting distribution, fix $x \in \mathbb{R}^d$ and put $\tau = \inf\{n \geq 0 : Y_n \in B + x\}$. Since the first visit must be at time τ,

$$\eta(B+x) = [\textstyle\sum_{n\geq\tau} \delta_{Y_n}](B+x)1\{\tau < \infty\}.$$

From the strong Markov property, for any n,

$$\mathbb{P}(\eta(B+x) \geq n) = \mathbb{E}(\mathbb{P}_{Y_\tau}(\eta(B+x) \geq n)1\{\tau < \infty\}). \tag{1}$$

If Y_n is a random walk, then for any $y \in B + x$,

$$\mathbb{P}_y(\eta(B+x) \geq n) = \mathbb{P}_0(\eta(B+x-y) \geq n) \leq \mathbb{P}_0(\eta(B-B) \geq n), \tag{2}$$

and plugging back into (1), we get $\eta(B+x) \leq^d \eta_0(B-B)$[1]. Applied to an increasing random walk W_n, with $B = [0, k]$, we get

$$\eta^W[x, x+k] \overset{d}{\leq} \eta_0^W[0, k]. \tag{3}$$

2. A pair of processes W_n and Z_n.

Suppose that W_n is an increasing random walk on $\mathbb{R}$, and that Z_n is an increasing Markov process on $\mathbb{R}$ such that $W_n \leq Z_n$ for all n.

We want to show that the average number of visits of Z_n to an interval is dominated by the same for W_n. We will develop a substitute (4) for the inequality (2), which is then plugged into (1) to give the desired result.

[1] See the proof of [K; Lemma 9.22]

For $y \in [x, x+k]$,

$$\begin{aligned}
\mathbb{P}(\eta^Z[x,x+k] \geq n \mid Z_0 = y) &\leq \mathbb{P}(\eta^Z[y,y+k] \geq n \mid Z_0 = y) \\
&\leq \mathbb{P}(\eta^W[y,y+k] \geq n \mid W_0 = y) \\
&= \mathbb{P}(\eta^W[0,k] \geq n \mid W_0 = 0). \qquad (4)
\end{aligned}$$

Plugging back into (1), with $B = [0, k]$, we obtain

$$\eta^Z[x, x+k] \overset{d}{\leq} \eta_0^W[0,k]. \qquad (5)$$

We now consider the second moment of the occupation measure. From (3) and (5), the squared number of visits (by both processes combined) to $[x, x+k]$ satisfies

$$\begin{aligned}
\mathbb{E}\Big(((\eta^Z + \eta^W)[x,x+k])^2\Big) &\leq 2\mathbb{E}\Big((\eta^Z[x,x+k])^2\Big) + 2\mathbb{E}\Big((\eta^W[x,x+k])^2\Big) \\
&\leq 4\mathbb{E}\Big((\eta_0^W[0,k])^2\Big). \qquad (6)
\end{aligned}$$

That this moment is finite can be seen as follows. When W_n starts at 0, we have $W_n = \xi_1 + \xi_2 + \cdots + \xi_n$ where the ξs are i.i.d. and $\mathbb{P}(\xi > 0) = 1$. Therefore

$$\mathbb{P}(\eta_0^W[0,k] > n) \leq \mathbb{P}(W_n \leq k) \leq e^k \mathbb{E}(e^{-W_n}) = e^k \left[\mathbb{E}(e^{-\xi})\right]^n,$$

so that

$$\mathbb{E}\left((\eta_0^W[0,k])^2\right) = \sum_{n=0}^{\infty}(2n+1)\mathbb{P}(\eta_0^W[0,k] > n) \leq \frac{e^k(1+\mathbb{E}(e^{-\xi}))}{(1-\mathbb{E}(e^{-\xi}))^2} \qquad (7).$$

The first coupling.

We begin by expressing the Markov chain X_n as a random dynamical system à la [K; Proposition 8.6]. Let $(U_n)_{n \geq 1}$ be a sequence of i.i.d. uniform $(0,1)$ random variables. Set $X_0 = 1$ and for $n \geq 1$ put $X_n = X_{n-1} + \lceil U_n X_{n-1} \rceil$. We define $Z_n = \log(X_n)$ so that

$$Z_n = \sum_{j=1}^{n} \log(1 + \lceil U_j X_{j-1} \rceil / X_{j-1}).$$

This suggests that the process Z_n is very close to the random walk

$$W_n = \sum_{j=1}^{n} \log(1 + U_j).$$

The next lemma will help us to bound the distance between Z_n and W_n.

Lemma. $\mathbb{E}(1/X_n^2) \leq (1/2)^n$.

Proof.

$$\mathbb{E}\left(\frac{1}{X_{n+1}^2} \;\Big|\; X_n\right) = \frac{1}{X_n} \sum_{j=X_n+1}^{2X_n} \frac{1}{j^2} \leq \frac{1}{2X_n^2}.$$

Taking expectations gives us $\mathbb{E}\left(1/X_{n+1}^2\right) \leq (1/2)\,\mathbb{E}\left(1/X_n^2\right)$, and the result follows, since $X_0 \geq 1$. $\square$

Here is a consequence of the lemma that we will need: First write

$$\left(\sum_{n\geq N} \frac{1}{X_n}\right)^2 \leq \left(\sum_{n\geq N} \frac{n^2}{X_n^2}\right)\left(\sum_{n\geq N} \frac{1}{n^2}\right),$$

and then take expectations to get

$$\mathbb{E}\left[\left(\sum_{n\geq N} \frac{1}{X_n}\right)^2\right] \leq \sum_{n\geq N} \frac{n^2}{2^n}\left(\frac{\pi^2}{6}\right) \to 0 \text{ as } N \to \infty. \tag{8}$$

We fix a smooth function f with support on (a,b), and $\varepsilon > 0$. The next coupling will depend on a, b, $\|f'\|_\infty$, and ε.

The second coupling. Define X_n and Z_n as in the first coupling, and run X_n up to time N, where N is to be determined. Define $W_n := Z_N + \sum_{j=N+1}^{n} \log(1+U_j)$, $n \geq N$, where the U_j's are the i.i.d. uniform $(0,1)$ random variables used to define X_n. Simple properties of the ceiling and the log function show that

$$W_n \leq Z_n \leq W_n + \sum_{j=N}^{n-1} \frac{1}{X_j},\; n \geq N.$$

Now we compare the random variable $(\theta_x\eta^Z)f = \sum_n f(Z_n - x)$ to the random walk version. Here the sum is over all n, but when $\log(N) - a \leq x$, then $Z_N \leq x + a$ and $f(Z_n - x) = 0$ for $n \leq N$. (We use the fact that $X_N \geq N$.) Hence, for $x \geq \log(N) - a$ we have

$$\left|(\theta_x\eta^Z)f - (\theta_x\eta^W)f\right| \;\leq\; \sum_{n\geq N} |f(Z_n - x) - f(W_n - x)|$$

$$
\begin{aligned}
&= \sum_{n\geq N} |f'(\xi_n)|(Z_n - W_n) \\
&\leq \|f'\|_\infty (\eta^Z + \eta^W)[a+x, b+x] \sum_{n\geq N} \frac{1}{X_n},
\end{aligned}
$$

where $W_n \leq \xi_n + x \leq Z_n$. The final inequality uses the fact that if both W_n and Z_n are outside the interval $[x+a, x+b]$, then $f'(\xi_n) = 0$. Taking expectations, and using Cauchy-Schwarz and (6), (8) we get

$$
\left|\mathbb{E}((\theta_x \eta^Z)f) - \mathbb{E}((\theta_x \eta^W)f)\right| \leq \|f'\|_\infty \sqrt{4\mathbb{E}((\eta_0^W[0, b-a])^2)}\, \varepsilon(N), \tag{9}
$$

where $\varepsilon(N) \to 0$ as $N \to \infty$. Choosing N large enough, we can make the right hand side of (9) less than ε. By the renewal theorem [K; Theorem 9.20 (ii)] we know that

$$
\lim_{x\to\infty} \mathbb{E}((\theta_x \eta^W)f) = \frac{1}{m} \int f(y)\, dy,
$$

where $m := \mathbb{E}(\log(1+U)) = 2\log(2) - 1$. Since ε is arbitrary, the result is also true for the process Z_n, that is,

$$
\lim_{x\to\infty} \mathbb{E}((\theta_x \eta^Z)f) = \frac{1}{m} \int f(y)\, dy. \tag{10}
$$

The number of visits of X_n to x. As before, we have the equation $p(x) = \sum_{y=\lceil x/2\rceil}^{x-1} p(y)/y$, which gives, for $z = \log(x)$,

$$
xp(x) = \mathbb{E}\left(\sum_{x/2\leq X_n < x} \frac{x}{X_n}\right) = \mathbb{E}\left(\sum_{z-\log(2)\leq Z_n < z} \exp(-[Z_n - z])\right).
$$

Thus, $xp(x) = \mathbb{E}((\theta_z \eta^Z)f)$ for the function $f(y) = 1_{[-\log(2),0)}(y)\exp(-y)$.

Applying (10) to smooth compactly supported functions ℓ, u such that $\ell \leq f \leq u$ and $\int (u(y) - \ell(y))\, dy$ is small, we conclude that

$$
\lim_{x\to\infty} xp(x) = \frac{1}{2\log(2)-1} \int_{-\log(2)}^{0} \exp(-y)\, dy = \frac{1}{2\log(2)-1}.
$$

Reference.

[K] *Foundations of Modern Probability, Second Edition.* Olav Kallenberg, Springer-Verlag 2002.