How far is the moon from the sun?

1 Question

Which is closer to the Sun; the Moon or the Earth? Since the Moon revolves around the Earth, the question is ambiguous. At a full Moon, for example, the Earth is closer. But at a new Moon, the Moon is closer.

Let us rephrase the question: On average, which is closer to the Sun; the Moon or the Earth? In this note, we will investigate this and other problems related to the path that the Moon traces out around the Sun.

2 The Model

Let's consider an ideal system where a planet moves uniformly in a circular orbit around its sun, while its moon moves uniformly in a circular orbit around the planet.

Placing the sun at the origin, and putting the planet-moon distance equal to one, the path of the moon around the sun is given by:

$$(x(t), y(t)) = (d\cos(t) + \cos(pt), d\sin(t) + \sin(pt)). \tag{1}$$

Here time is scaled so that the planet makes one orbit around the sun as t moves from 0 to 2π, that is, one year passes. Also, $d \geq 0$ is the sun-planet distance and p the angular velocity (per year) of the moon in its orbit around the planet.

We are interested in the path that the moon makes around the sun, especially in its distance from the sun. The moon's path was analysed in [1, 2], where it is shown that if the planet is far from the sun (d large) and the moon slow (p small), then the resulting path is convex. For Earth ($d = 400, p = 13$) this counter-intuitive result is, in fact, true.

The next four figures illustrate some of the possibilities for the moon's path. In these pictures, we have always taken $d = 2$ and, for comparison purposes, have drawn the planet's orbit around the sun in bold.

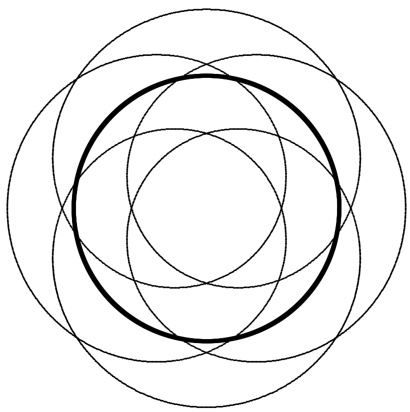

Figure 1. Four months every five years ($p = 1/5$)

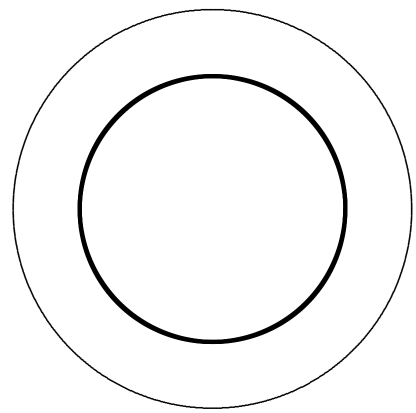

Figure 2. Eternal full moon ($p = 1$)

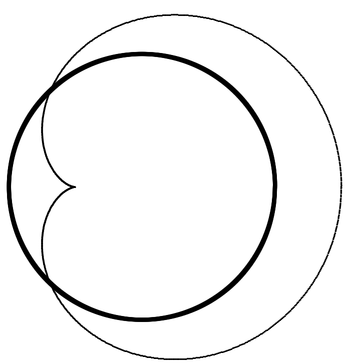

Figure 3. A one-month year ($p = 2$)

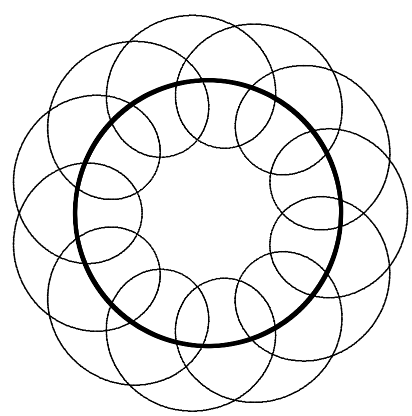

Figure 4. A twelve-month year ($p = 13$)

3 Periodic Behavior

The distance of the moon from the sun at time t is $\sqrt{x(t)^2 + y(t)^2}$. At time $t = 0$, for instance, we have a full moon, that is, the moon is at the maximum distance $d + 1$ from the sun.

Using (1), expanding the squares, and using the formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, we may rewrite this as:

$$\sqrt{x(t)^2 + y(t)^2} = \sqrt{d^2 + 2d\cos((p-1)t) + 1}. \tag{2}$$

From (2) we see that, barring the case $p = 1$, the next full moon after $t = 0$ will occur at $t_p := 2\pi/|p-1|$. The process repeats itself starting from the new location, so that the piece of the path $\{(x(t), y(t)) : t_p \leq t \leq 2t_p\}$, is a copy of the piece of path $\{(x(t), y(t)) : 0 \leq t \leq t_p\}$ rotated through t_p radians.

If p is a rational number, then the moon will eventually return to its original position, and the path around the sun will form a closed curve. Figures 1–4 illustrate this. However, if p is an irrational number, then the moon will never return to its original position, and the path will densely fill the ring of points whose distance from the sun lies between $d - 1$ and $d + 1$. Ergodic theory says that the long run distribution of $(x(t), y(t))$ is the same as the probability distribution of a randomly chosen moon. This "random moon" is obtained by first fixing the planet at a random spot chosen from its orbit, and then picking a random spot on the moon's orbit. This is illustrated by Figures 5 and 6.

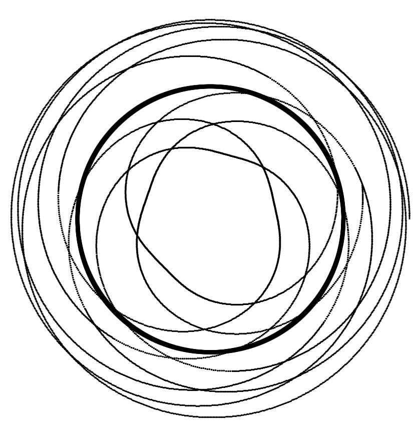

Figure 5. The moon's path for irrational p

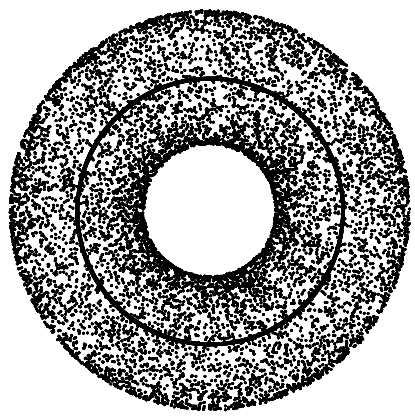

Figure 6. 10000 random points on the moon's path for irrational p

4 Sun-Moon Distance

Always ignoring the case $p = 1$, when we look at the different, complex, sometimes chaotic, paths that the moon may follow around the sun, it is remarkable that, for fixed d, the sun-moon distance always has the same statistical properties. This means that questions like "What is the average sun-moon distance?" or "What percentage of time does the moon spend inside the planet's orbit?" have the same answer for all the systems found in Figures 1, 3–6.

Mathematically, this can easily be seen from equation (2). Intuitively, we can understand this by reflecting on the fact that the distance from the sun to the moon depends only on the planet-moon system. The position of the planet in its orbit is irrelevant, to understand sun-moon distance we may as well consider the motion of the moon relative to the planet. This is simply a circle, of course.

The average distance of the moon to the sun is therefore

$$\text{Average}(d) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{d^2 + 1 + 2d\cos(t)}\, dt, \tag{3}$$

independently of p. This may be expressed in terms of complete elliptic integrals of the second kind. In Maple's notation this gives

$$\text{Average}(d) := \frac{2(d+1)\,\text{EllipticE}\left(\dfrac{2\sqrt{d}}{d+1}\right)}{\pi}. \tag{4}$$

Using Maple to solve this integral numerically gives, for $d = 2$, Average(2) = 2.12709, while for the real "Sun-Earth-Moon" system we get Average(400) = 400.00062.

Finally, to find the proportion of time that the moon spends inside the planet's orbit, we imagine fixing both the sun and the planet. What we want amounts to the proportion of the moon's orbit inside the planet's orbit, which we find using simple geometry. The two isosceles triangles in Figure 7 each have two sides of length d with a third side of length 1. We conclude that $\alpha = 2\arcsin(1/2d)$, and the proportion we want is $2\beta/2\pi = (\pi - \alpha)/2\pi$, that is,

$$\text{Inside}(d) := \frac{1}{2} - \frac{\arcsin(1/2d)}{\pi}. \tag{5}$$

For $d = 2$, we have Inside(2) = .41957, and for the real "Sun-Earth-Moon" system we get Inside(400) = .49960.

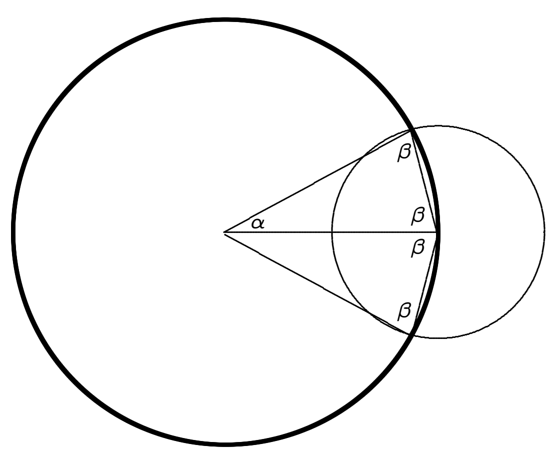

Figure 7. The moon's orbit

Results (3) and (5) show that for small d the moon is further from the sun than the planet well over 50% of the time, and its average distance is significantly larger than d. As d gets bigger, the moon's path is very close to a circle of radius d, and the moon spends almost exactly 50% of its time inside the planet's orbit, and 50% outside.

References

[1] Helmer Aslaksen, The Orbit of the Moon around the Sun is Convex! Webpage at `www.math.nus.edu.sg/aslaksen/teaching/convex.html`, accessed September 23 2003.

[2] Noah Samuel Brannen, The Sun, the Moon, and Convexity, *The College Mathematics Journal* **32** (2001) 268–272.