Does the Markov property only depend on finite dimensional distributions?

Let I be a subset of the real line and $(X_t)_{t\in I}$ a stochastic process on $(\Omega, \mathcal{F}, P)$ taking values in $(E, \mathcal{E})$. For $t \in I$, define sub-σ-fields

$$\mathcal{F}_{\leq t} := \sigma\{X_r : r \in I, r \leq t\}, \quad \mathcal{F}_{\geq t} := \sigma\{X_r : r \in I, r \geq t\}.$$

We say that (X_t) is a Markov process if for $t \in I$, $A \in \mathcal{F}_{\leq t}$, and $B \in \mathcal{F}_{\geq t}$, we have

$$P(A \cap B \mid X_t) = P(A \mid X_t)\, P(B \mid X_t), \tag{M}$$

where the equation is P-almost sure.

Lemma: If $(Y_t)_{t\in I}$ has the same distribution as $(X_t)_{t\in I}$, then (Y_t) is also Markov.

1. In equation (M), by fixing A and using the Monotone Class Theorem on B, and vice versa, we see that $(X_t)_{t\in I}$ is Markov if and only if $(X_t)_{t\in F}$ is Markov for every finite $F \subseteq I$.

2. Without loss of generality, we assume that I is finite. Define $L_t = \{i \in I : i \leq t\}$ and $R_t = \{i \in I : i \geq t\}$. The Markov property is equivalent to

$$\mathbb{E}\left(\prod_{i\in I} 1_{E_i}(X_i) \,\middle|\, X_t\right) = \mathbb{E}\left(\prod_{i\in L_t} 1_{E_i}(X_i) \,\middle|\, X_t\right) \mathbb{E}\left(\prod_{i\in R_t} 1_{E_i}(X_i) \,\middle|\, X_t\right), \tag{M^*}$$

   for any collection $(E_i)_{i\in I}$ in $\mathcal{E}$.

3. Suppose that $(W, X) : (\Omega, P) \to \mathbb{R} \times E$ and $(Z, Y) : (\Omega', Q) \to \mathbb{R} \times E$ are identically distributed. If $E(W \mid X) = \Psi(X)$ for some Borel function Ψ, then $\Psi(Y) = E(Z \mid Y)$. To prove this, take any $E' \in \mathcal{E}$ and calculate:

$$\int_{\Omega'} \Psi(Y) 1_{(Y\in E')}\, dQ = \int_{\Omega} \Psi(X) 1_{(X\in E')}\, dP = \int_{\Omega} W 1_{(X\in E')}\, dP = \int_{\Omega'} Z 1_{(Y\in E')}\, dQ.$$

4. Write

$$\begin{aligned}
\mathbb{E}\left(\prod_{i\in I} 1_{E_i}(X_i) \,\middle|\, X_t\right) &= F(X_t)\\
\mathbb{E}\left(\prod_{i\in L_t} 1_{E_i}(X_i) \,\middle|\, X_t\right) &= L(X_t)\\
\mathbb{E}\left(\prod_{i\in R_t} 1_{E_i}(X_i) \,\middle|\, X_t\right) &= R(X_t)
\end{aligned}$$

   for Borel functions F, L, R. Equation (M^*) becomes $F(e) = L(e)R(e)$ for $\mathcal{L}(X_t)$-almost every $e \in E$. Since $(Y_t)_{t\in I}$ and $(X_t)_{t\in I}$ have the same laws, using the result in point 3 we see that the equation (M^*) also holds for the process (Y_t).