The law of large numbers and the law of the iterated logarithm for infinite dimensional interacting diffusion processes

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Abstract

The classical Dirichlet form given by the intrinsic gradient on $\Gamma_{\mathbb{R}^d}$ is associated with a Markov process consisting of a countable family of interacting diffusions. By considering each diffusion as a particle with unit mass, the randomly evolving configuration can be thought of as a Radon measure valued diffusion.

The quasi-sure analysis of Dirichlet forms is used to find exceptional sets of configurations for this Markov process. We consider large scale properties of the configuration and show that, for quite general measures, the process never hits those unusual configurations that violate the law of large numbers. Furthermore, for certain Gibbs measures, which model random particles in $\mathbb{R}^d$ that interact via a potential function, we show, for $d = 1, 2$, that the process never hits those unusual configurations that violate the law of the iterated logarithm.

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1 Introduction

In a pair [3, 4] of fundamental papers in 1998, Albeverio, Kondratiev, and Röckner began a study of analysis and geometry on configuration space.

(This work had been previously announced in [1, 2] and anticipated in the 1996 papers by Osada [11] and Yoshida [23].) One of the features of [3, 4] was the construction of a configuration-valued Markov process using a Dirichlet form based on an intrinsic gradient defined on configuration space.

In this paper, we look at some sample path properties of the Albeverio-Kondratiev-Röckner process in the case where the underlying manifold is d-dimensional Euclidean space. For quite general stationary measures for the process, we will show that the associated Markov process never hits configurations that violate the law of large numbers. Furthermore, for a Ruelle measure μ with small activity parameter z the law of the iterated logarithm (Proposition 3) holds for μ-almost every configuration $\gamma \in \Gamma_{\mathbb{R}^d}$. In dimensions d less than or equal to 2, we also show that the associated Markov process never hits configurations that violate the law of the iterated logarithm. We do so by proving that these sets of unusual configurations are exceptional sets for the Dirichlet form.

2 Classical Dirichlet forms on configuration space

In this section we recall some of the basic definitions and properties about the classical Dirichlet forms on configuration spaces. For more detailed definitions and fuller explanations we refer the reader to [3, 4, 7, 13].

The space of locally finite configurations in $\mathbb{R}^d$ is defined by

$$\Gamma_{\mathbb{R}^d} := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for every compact } K\}.$$

A configuration γ will be identified with the Radon measure $\sum_{x\in\gamma} \varepsilon_x$. The space $\Gamma_{\mathbb{R}^d}$ will be given the topology of vague convergence of measures, and measures on $\Gamma_{\mathbb{R}^d}$ are defined on the corresponding Borel sets $\mathcal{B}(\Gamma_{\mathbb{R}^d})$.

For $f \in C_0(\mathbb{R}^d)$ we let $\langle f, \gamma\rangle$ be the integral of f with respect to the measure γ, that is, $\langle f, \gamma\rangle = \sum_{x\in\gamma} f(x)$. Define

$$\mathcal{F}C_b^\infty := \{\ u : u(\gamma) = g(\langle f_1, \gamma\rangle, \langle f_2, \gamma\rangle, \ldots, \langle f_n, \gamma\rangle) \\ \text{for some } f_i \in C_0^\infty(\mathbb{R}^d) \text{ and } g \in C_b^\infty(\mathbb{R}^n)\ \}.$$

For $u \in \mathcal{F}C_b^\infty$, we define the gradient $\nabla^\Gamma u$ at the point $\gamma \in \Gamma_{\mathbb{R}^d}$ as an element of the "tangent space" $T_\gamma(\Gamma_{\mathbb{R}^d}) := L^2(\mathbb{R}^d \to \mathbb{R}^d; \gamma)$ by the formula

$$\left(\nabla^\Gamma u\right)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(\langle f_1, \gamma\rangle, \langle f_2, \gamma\rangle, \ldots, \langle f_n, \gamma\rangle\right) \nabla f_i(x).$$

Here ∇ refers to the usual gradient on $\mathbb{R}^d$. It is not hard to prove that $\nabla^\Gamma u$ is well-defined, even though the representation of u as a cylinder function is not unique.

Definition 1 *For $u, v \in \mathcal{F}C_b^\infty$ define the square field $S(u, v)$ as the real-valued function on $\Gamma_{\mathbb{R}^d}$ given by*

$$\begin{aligned} S(u,v)(\gamma) &:= \langle \nabla^\Gamma u, \nabla^\Gamma v\rangle_{T_\gamma(\Gamma_{\mathbb{R}^d})} \\ &= \int_{\mathbb{R}^d} \langle (\nabla^\Gamma u)(\gamma; x), (\nabla^\Gamma v)(\gamma; x)\rangle_{\mathbb{R}^d}\, \gamma(dx). \end{aligned}$$

We will often use the abbreviation $S(u) := S(u, u)$.

In this section we fix a probability measure μ on $\Gamma_{\mathbb{R}^d}$ with Radon mean, that is,

$$\int_{\Gamma_{\mathbb{R}^d}} \gamma(K)\mu(d\gamma) < \infty \text{ for all compact } K \subset \mathbb{R}^d. \qquad (\mu.1)$$

Furthermore, assume that

$$\nabla^\Gamma u = \nabla^\Gamma v \ \mu\text{-a.e. if } u, v \in \mathcal{F}C_b^\infty \text{ such that } u = v \ \mu\text{-a.e.} \qquad (\mu.2)$$

Definition 2 *For $u, v \in \mathcal{F}C_b^\infty$ define the pre-Dirichlet form by*

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\Gamma_{\mathbb{R}^d}} S(u, v)(\gamma)\, \mu(d\gamma).$$

Let $\mathcal{F}C_b^{\infty,\mu}$ denote the set of μ-equivalence classes determined by $\mathcal{F}C_b^\infty$. Suppose that

$$(\mathcal{E}, \mathcal{F}C_b^{\infty,\mu}) \text{ is closable on } L^2(\Gamma_{\mathbb{R}^d}; \mu), \qquad (\mu.3)$$

and denote the closure by $(\mathcal{E}, D(\mathcal{E}))$. The conditions $(\mu.1)$, $(\mu.2)$, and $(\mu.3)$ are the minimal requirements on μ so that we can define the classical Dirichlet form on configuration space. See Section 5 of this paper, as well as [3, 4, 7, 13] for concrete examples of Gibbs measures that satisfy these conditions.

We note that the square field operator S extends to the full domain $D(\mathcal{E})$ so that $\mathcal{E}(u,v) = 1/2 \int S(u,v)\, d\mu$ for all $u, v \in D(\mathcal{E})$. In addition, we will frequently use the standard result for square field Dirichlet forms that says if $u, v \in D(\mathcal{E})$, then $u \vee v, u \wedge v \in D(\mathcal{E})$ and

$$S(u \vee v) \vee S(u \wedge v) \leq S(u) \vee S(v). \tag{1}$$

Even under these minimal conditions on μ it follows that $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric, quasi-regular, and local Dirichlet form. The quasi-regularity and locality of $(\mathcal{E}, D(\mathcal{E}))$ has been proven for certain cases by Yoshida [23], and in general by Ma and Röckner [7]. However, the proof of quasi-regularity requires extending the state space to the set of multiple configurations:

$$\bar{\Gamma}_{\mathbb{R}^d} := \{\mathbb{Z}_+ \cup \{+\infty\}\text{-valued Radon measures on } \mathbb{R}^d\}.$$

Since $\Gamma_{\mathbb{R}^d} \subset \bar{\Gamma}_{\mathbb{R}^d}$ and $\mathcal{B}(\bar{\Gamma}_{\mathbb{R}^d}) \cap \Gamma_{\mathbb{R}^d} = \mathcal{B}(\Gamma_{\mathbb{R}^d})$, we can consider μ as a measure on $(\bar{\Gamma}_{\mathbb{R}^d}, \mathcal{B}(\bar{\Gamma}_{\mathbb{R}^d}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\bar{\Gamma}_{\mathbb{R}^d}; \mu)$. The associated strong Markov process $(X_t)_{t\geq 0}$ has vaguely continuous sample paths since $(\mathcal{E}, D(\mathcal{E}))$ is a local form [6, Chapter V, Theorem 1.11]. Note that by [15, 8], if $d \geq 2$, the set $\bar{\Gamma}_{\mathbb{R}^d} \setminus \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$-exceptional for a large class of measures μ on $\Gamma_{\mathbb{R}^d}$.

We recall the following results from Dirichlet form theory. Lemma 1 (see [19]) is used to prove that certain sets are $\mathcal{E}$-exceptional while Lemma 2 gives us the interpretation in terms of the sample paths of $(X_t)_{t\geq 0}$.

Lemma 1 *Let $u_n \in D(\mathcal{E})$ be a sequence of $\mathcal{E}$-quasi-continuous functions with $\sup_n \mathcal{E}(u_n, u_n) < \infty$ and $u_n \to u$ pointwise. Then u is an $\mathcal{E}$-quasi-continuous function, in particular, for μ-almost every $\gamma \in \bar{\Gamma}_{\mathbb{R}^d}$,*

$$P_\gamma\left(t \to u(X_t) \text{ is continuous }\right) = 1.$$

If u is μ-square integrable, then $u \in D(\mathcal{E})$.

Lemma 2 *A set $N \in \mathcal{B}(\bar{\Gamma}_{\mathbb{R}^d})$ is $\mathcal{E}$-exceptional if and only if, for μ-almost every $\gamma \in \bar{\Gamma}_{\mathbb{R}^d}$,*

$$P_\gamma\left(X_t \in N \text{ for some } 0 \leq t < \infty\right) = 0.$$

We sometimes refer to proofs of exceptionality as *capacitary* since N is $\mathcal{E}$-exceptional if and only if $\mathrm{Cap}(N) = 0$ for a suitably defined capacity Cap on $\bar{\Gamma}_{\mathbb{R}^d}$ [6, Chapter III, Theorem 2.11].

3 Law of large numbers on configuration space

For $l > 0$ let C_l be the cube $(-(l+1/2), l+1/2]^d$, and for every $r \in \mathbb{Z}^d$ let $Q_r = r + (-1/2, 1/2]^d$. Denote the Lebesgue measure of any Borel subset G of $\mathbb{R}^d$ by $|G|$. Define

$$LLN := \left\{ \gamma \in \bar{\Gamma}_{\mathbb{R}^d} : \lim_{n \in \mathbb{N}} \frac{\gamma(C_n)}{|C_n|} \text{ exists} \right\} = \left\{ \gamma \in \bar{\Gamma}_{\mathbb{R}^d} : \lim_{l \in \mathbb{R}_+} \frac{\gamma(C_l)}{|C_l|} \text{ exists} \right\}.$$

Our goal is to show that the complement of this set has zero capacity.

In this section we fix a probability measure μ on $\bar{\Gamma}_{\mathbb{R}^d}$ that satisfies the following conditions. In Section 5, we will see a class of Gibbs measures that satisfy these conditions.

Conditions on μ

(μ.4) The measure μ satisfies (μ.1), (μ.2), and (μ.3) from Section 2.

(μ.5) The measure μ is translation invariant.

Proposition 1 recalls the well-known fixed time law of large numbers ([9, Proposition 4.23]), while Proposition 2 gives the full capacitary version.

Proposition 1 *The law of large numbers holds for μ-almost every γ, that is, $\mu(LLN^c) = 0$.*

Proof. Define the Abelian group $(T_r)_{r \in \mathbb{Z}^d}$ of automorphisms on $\bar{\Gamma}_{\mathbb{R}^d}$ by $T_r \gamma(G) = \gamma(G - r)$ for any bounded Borel G. The family $\{\gamma(G) : G \in \mathcal{B}(\mathbb{R}^d),\ \text{bounded}\}$ is trivially an additive covariant spatial process in the sense of [9]. Therefore, by [9, Proposition 4.23],

$$\lim_{n \to \infty} \frac{\gamma(C_n)}{|C_n|} = E_\mu(\gamma(Q_0) \mid \mathcal{H}),\ \ \mu\text{−a.e.},$$

where $\mathcal{H}$ denotes the σ-algebra of $(T_r)_{r \in \mathbb{Z}^d}$-invariant sets in $\mathcal{B}(\bar{\Gamma}_{\mathbb{R}^d})$. □

Proposition 2 *The set LLN^c is $\mathcal{E}$-exceptional.*

Proof. For every $n \geq 3$, let ψ_n be a smooth function on $\mathbb{R}$ satisfying $1_{(-(n-3/2), n-3/2]} \leq \psi_n \leq 1_{(-(n+1/2), n+1/2]}$, and $|\psi_n'| \leq 1$. Let ϕ be a strictly increasing, smooth, bounded function on $\mathbb{R}$ with $\sup_x |\phi'(x)| \leq 1$. Define a continuous element of $D(\mathcal{E})$ by

$$u_n(\gamma) := \phi(\langle \prod_{i=1}^d \psi_n(x_i), \gamma\rangle / |C_n|).$$

Taking the limsup through the inequality

$$\phi\left(\frac{|C_{n-2}|}{|C_n|}\,\frac{\gamma(C_{n-2})}{|C_{n-2}|}\right) \leq u_n(\gamma) \leq \phi\left(\frac{\gamma(C_n)}{|C_n|}\right),$$

we find that $u(\gamma) := \limsup_n u_n(\gamma)$ is equal to $\limsup_n \phi(\gamma(C_n)/|C_n|) = \phi(\limsup_n \gamma(C_n)/|C_n|)$.

We want to apply Lemma 1 to show that u is an $\mathcal{E}$-quasi-continuous function, so we begin by bounding the square field of u_n, which gives

$$S(u_n)(\gamma) \leq d\,\frac{\gamma(C_n)}{|C_n|^2}. \tag{2}$$

Since C_n is an increasing sequence, there exists a sequence $(r_m)_{m\in\mathbb{N}}$ in $\mathbb{Z}^d$ so $C_n = \cup_{m=1}^{|C_n|} Q_{r_m}$ for $n \in \mathbb{N}$. Then

$$\frac{\chi_{C_n}}{|C_n|^2} \leq \sum_{m=1}^{|C_n|} \frac{\chi_{Q_{r_m}}}{m^2} \leq \sum_{m=1}^{\infty} \frac{\chi_{Q_{r_m}}}{m^2},$$

so that

$$\begin{aligned}
\int_{\bar{\Gamma}_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \frac{\gamma(C_n)}{|C_n|^2}\,\mu(d\gamma) &= \int_{\bar{\Gamma}_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \frac{\chi_{C_n}(x)}{|C_n|^2}\,\gamma(dx)\,\mu(d\gamma) \\
&\leq \int_{\bar{\Gamma}_{\mathbb{R}^d}} \int_{\mathbb{R}^d} \sup_{n\in\mathbb{N}} \frac{\chi_{C_n}(x)}{|C_n|^2}\,\gamma(dx)\,\mu(d\gamma) \\
&\leq \int_{\bar{\Gamma}_{\mathbb{R}^d}} \int_{\mathbb{R}^d} \sum_{m=1}^{\infty} \frac{\chi_{Q_{r_m}}(x)}{m^2}\,\gamma(dx)\,\mu(d\gamma) \\
&= \int_{\bar{\Gamma}_{\mathbb{R}^d}} \sum_{m=1}^{\infty} \frac{\gamma(Q_{r_m})}{m^2}\,\mu(d\gamma) \\
&= \sum_{m=1}^{\infty} \int_{\bar{\Gamma}_{\mathbb{R}^d}} \frac{\gamma(Q_{r_m})}{m^2}\,\mu(d\gamma) \\
&= \sum_{m=1}^{\infty} \frac{1}{m^2} E_\mu(\gamma(Q_0)) \\
&< \infty.
\end{aligned}$$

Let us denote the random variable $X^*(\gamma) := \sup_{n\in\mathbb{N}} \gamma(C_n)/|C_n|^2$. For fixed $n \in \mathbb{N}$ let $A_j = \{n, n+1, \ldots, n+j\}$ so that

$$\sup_{k\geq n} u_k(\gamma) = \sup_j \sup_{k\in A_j} u_k(\gamma).$$

Now for each $j \in \mathbb{N}$, $\sup_{k\in A_j} u_k \in D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. Repeated use of (1) combined with the bound (2) gives $S(\sup_{k\in A_j} u_k) \leq d\, X^*$, and so

$$\sup_j \mathcal{E}\left(\sup_{k\in A_j} u_k, \sup_{k\in A_j} u_k\right) \leq d \int_{\bar{\Gamma}_{\mathbb{R}^d}} X^*(\gamma)\, \mu(d\gamma) < \infty.$$

Applying Lemma 1, we see that the pointwise limit $\sup_{k\geq n} u_k$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. In addition, the bound for the square field also carries over; $S(\sup_{k\geq n} u_k) \leq dX^*$. Applying the same argument to the decreasing sequence $(\sup_{k\geq n} u_k)_{n\in\mathbb{N}}$, we find that the pointwise limit u belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous.

A parallel argument shows that $v(\gamma) := \liminf_{n\to\infty} \phi(\gamma(C_n)/|C_n|)$ is also $\mathcal{E}$-quasi-continuous. Since, by Proposition 1, the two $\mathcal{E}$-quasi-continuous functions u and v agree μ-almost everywhere, they must agree except on an $\mathcal{E}$-exceptional set [6, Chapter IV, Proposition 3.3]. Applying the inverse function ϕ^{-1} shows that the limsup and liminf agree except on an $\mathcal{E}$-exceptional set, which gives us the result. □

Remark. We only used the translation invariant property of μ to get Proposition 1 and to ensure that $\sup_{r\in\mathbb{Z}^d} E_\mu(\gamma(Q_r)) < \infty$. If μ satisfies (μ.4) and $\sup_{r\in\mathbb{Z}^d} E_\mu(\gamma(Q_r)) < \infty$, then we can drop the translation invariance and Proposition 1, and easily modify the proof above (along the lines of [19, Proposition 5]) to conclude that $\limsup_n X_t(C_n)/|C_n|$ is almost surely constant in t along sample paths. That is, for μ-almost every $\gamma \in \bar{\Gamma}_{\mathbb{R}^d}$, we have

$$P_\gamma\left\{\limsup_n \frac{X_0(C_n)}{|C_n|} = \limsup_n \frac{X_t(C_n)}{|C_n|} \text{ for all } t \geq 0\right\} = 1,$$

and of course, the parallel result with the liminf.

Roughly speaking, this result shows that the large scale density of particles does not change over time, so that the law of large numbers holds (or fails to hold) along sample paths exactly as at time zero.

In particular, this argument also gives the capacitary version of the law of large numbers for (mixed) Poisson point processes in $\mathbb{R}^d$. This improves Proposition 5 of [18] by removing the condition $\int_{\mathbb{R}_+} z \log^+(z)\, \lambda(dz) < \infty$.

4 Law of the iterated logarithm on configuration space

As in Section 3, let $C_n = (-(n+1/2), n+1/2]^d$ be the centered cube with volume $(2n+1)^d$, and set $\mu_n = E_\mu(\gamma(C_n))$, $\sigma_n^2 = \mathrm{Var}_\mu(\gamma(C_n))$, and $\chi_n := (2\sigma_n^2 \log\log \sigma_n^2)^{1/2}$. Letting $S_n = \gamma(C_n) - \mu_n$, we define

$$LIL := \left\{ \gamma \in \bar{\Gamma}_{\mathbb{R}^d} : \limsup_n \frac{S_n}{\chi_n} = 1 \text{ and } \liminf_n \frac{S_n}{\chi_n} = -1 \right\}.$$

Our goal is to show that the complement of this set has zero capacity.

We fix a probability measure μ on $\bar{\Gamma}_{\mathbb{R}^d}$ that satisfies the following conditions. In Section 5, we will see a class of Gibbs measures that satisfy these conditions.

Conditions on μ

(μ.4) The measure μ satisfies (μ.1), (μ.2), and (μ.3) from Section 2.

(μ.5) The measure μ is translation invariant.

(μ.6) There exist constants $a, b > 0$ so that $0 < a \leq \sigma_n^2/|C_n| \leq b < \infty$.

(μ.7) For bounded regions $\Lambda_i, (i = 1, 2)$ and $\Psi_i \in \mathcal{F}(\Lambda_i)$ that are square integrable, there are constants $\alpha, c > 0$, depending only on z,ϕ, such that

$$|\mathrm{Corr}_\mu(\Psi_1, \Psi_2)| \leq \min\{1, c\, |\Lambda_1^R|\, e^{-\alpha d(\Lambda_1, \Lambda_2)}\},$$

where $d(\Lambda_1, \Lambda_2) := \inf\{\|x - y\| : x \in \Lambda_1, y \in \Lambda_2\}$, and $\Lambda^R := \{x \in \mathbb{R}^d : d(x, \Lambda) \leq R\}$.

As with the law of large numbers, the proof of the law of the iterated logarithm begins with the fixed time result, that is, we prove $\mu(LIL^c) = 0$. The proof is based on the methods used in [10, 24], while replacing the strong mixing condition there by the exponential mixing condition in (μ.7). The full proof of the fixed time law of the iterated logarithm appears in a companion article [20].

Remark. The inequalities in (μ.6) and the exponential bound in (μ.7) are used in [20, Lemma 3] to prove that $\sigma_n^2/|C_n| \to \sigma^2$ for some $0 < \sigma^2 < \infty$ as $n \to \infty$. This means that the set LIL is unchanged if we replace χ_n in the definition by $\chi_n = (2\sigma^2|C_n|\log\log|C_n|)^{1/2}$. From now on, we will use this new definition of χ_n.

Proposition 3 *The law of the iterated logarithm holds for almost every* γ*, that is,* $\mu(LIL^c) = 0$.

Proof. Setting $x_i = \gamma(Q_i) - \rho$ for $i \in \mathbb{Z}^d$, the law of the iterated logarithm is Proposition 1 of [20].

Lemma 3 *(cf. [24, Lemma 1]) Let* $(r_j)_{j\in\mathbb{N}}$ *be a sequence in* $\mathbb{Z}^d$ *so that* $r_i \neq r_j$ *if* $i \neq j$. *Suppose* $(\eta_j)_{j\in\mathbb{N}}$ *are mean zero, square integrable random variables so that* $\eta_j \in \mathcal{F}(Q_{r_j})$ *for all* $j \in \mathbb{N}$. *Suppose also that* $M := \sup_j \mathrm{Var}_\mu(\eta_j) < \infty$. *Then there is a constant* $K < \infty$ *depending only on* z,ϕ*, and the dimension* d*, so that for any* n*,*

$$\mathrm{Var}_\mu\left(\sum_{j=1}^{n}\eta_j\right) \leq K\,M\,n, \tag{3}$$

and

$$E_\mu\left(\max_{m\leq n}(\eta_1+\cdots+\eta_m)^2\right) \leq K\,M\,n\,(\log_2 2n)^2. \tag{4}$$

Proof. To prove (3), it suffices to show that

$$K := \sup_i \sum_{j=1}^{\infty}|\mathrm{Corr}_\mu(\eta_i,\eta_j)| < \infty.$$

We will apply (μ.7) with Q_{r_i} and Q_{r_j} in place of Λ_1 and Λ_2. Since $|Q_r^R|$ doesn't depend on r, we can adjust the constants α and c (they now depend on d) to obtain $|\mathrm{Corr}_\mu(\eta_i,\eta_j)| \leq c\,e^{-\alpha|r_i-r_j|_\infty}$. Since the r_j's are distinct, we get

$$\begin{aligned}\sum_{j=1}^{\infty}|\mathrm{Corr}_\mu(\eta_i,\eta_j)| &\leq \sum_{j=1}^{\infty} c\,e^{-\alpha|r_i-r_j|_\infty}\\ &\leq \sum_{N=0}^{\infty} c\,e^{-\alpha N}\#\{j : |r_i-r_j| \leq N\}\\ &\leq \sum_{N=0}^{\infty} c\,e^{-\alpha N}(2N+1)^d < \infty.\end{aligned}$$

Since this sum is finite and independent of i, we get (3). The bound (4) follows directly from [21, Corollary A3.1] (with $\delta = 1$). □

Proposition 4 *In dimensions $d \leq 2$, the set LIL^c is $\mathcal{E}$-exceptional.*

Proof. For every $n \geq 3$, let ψ_n be a smooth function on $\mathbb{R}$ satisfying $1_{(-(n-3/2),n-3/2]} \leq \psi_n \leq 1_{(-(n+1/2),n+1/2]}$, and $|\psi_n'| \leq 1$. Let ϕ be a strictly increasing, smooth, bounded function on $\mathbb{R}$ with $\sup_x |\phi'(x)| \leq 1$. Define a continuous element of $D(\mathcal{E})$ by

$$u_n(\gamma) := \phi\left(\frac{\langle \prod_{i=1}^d \psi_n(x_i), \gamma\rangle - \rho|C_n|}{\chi_n}\right).$$

Taking the limsup through the inequality

$$\phi\left(\frac{S_{n-2}}{\chi_{n-2}}\,\frac{\chi_{n-2}}{\chi_n} - \frac{\rho|C_n \setminus C_{n-2}|}{\chi_n}\right) \leq u_n(\gamma) \leq \phi\left(\frac{S_n}{\chi_n}\right),$$

we find that $u(\gamma) := \limsup_n u_n(\gamma)$ is equal to $\limsup_n \phi(S_n/\chi_n) = \phi(\limsup_n S_n/\chi_n)$. Note that $|C_n \setminus C_{n-2}|$ is of order n^{d-1} while χ_n is of order $n^{d/2}\log\log(n)$, so that the condition $d \leq 2$ is needed to ensure that $|C_n \setminus C_{n-2}|/\chi_n \to 0$ as $n \to \infty$.

We want to apply Lemma 1 to show that u is an $\mathcal{E}$-quasi-continuous function, so we begin by bounding the square field of u_n, which gives

$$S(u_n)(\gamma) \leq d\,\frac{\gamma(C_n \setminus C_{n-2})}{\chi_n^2}. \tag{5}$$

Now

$$\begin{aligned}
\frac{\gamma(C_n \setminus C_{n-2})}{\chi_n^2} &\leq \frac{\gamma(C_n \setminus C_{n-2}) - \rho|C_n \setminus C_{n-2}|}{\chi_n^2} + \frac{\rho|C_n \setminus C_{n-2}|}{\chi_n^2} \\
&\leq \frac{|\gamma(C_n) - \rho|C_n||}{\chi_n^2} + \frac{|\gamma(C_{n-2}) - \rho|C_{n-2}||}{\chi_n^2} + \frac{\rho\,|C_n \setminus C_{n-2}|}{\chi_n^2}. \\
&\leq \frac{1}{2\sigma^2}\left(\frac{|\gamma(C_n) - \rho|C_n||}{|C_n|} + \frac{|\gamma(C_{n-2}) - \rho|C_{n-2}||}{|C_{n-2}|} + \rho\right).
\end{aligned}$$

Together with (5) this gives us

$$\sup_{n\geq 3}\Big(S(u_n)(\gamma)\Big) \leq c\left(1 + \sup_{n\geq 1}\frac{|\gamma(C_n) - \rho|C_n||}{|C_n|}\right). \tag{6}$$

As in the proof of Proposition 2, let $(r_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{Z}^d$ so $C_n = \cup_{m=1}^{|C_n|} Q_{r_m}$ for $n \in \mathbb{N}$. Denote $\eta_m(\gamma) = \gamma(Q_{r_m}) - \rho$ for $m \in \mathbb{N}$. By applying Lemma 3, we have

$$\begin{aligned}
\int_{\bar\Gamma_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \frac{|\gamma(C_n) - \rho|C_n||}{|C_n|}\, \mu(d\gamma) &= \int_{\bar\Gamma_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \frac{|\eta_1 + \cdots + \eta_{|C_n|}|}{|C_n|}\, \mu(d\gamma) \\
&\le \int_{\bar\Gamma_{\mathbb{R}^d}} \sup_{m\in\mathbb{N}} \frac{|\eta_1 + \cdots + \eta_m|}{m}\, \mu(d\gamma) \\
&= \lim_{k\to\infty} \int_{\bar\Gamma_{\mathbb{R}^d}} \max_{m\le 2^k} \frac{|\eta_1 + \cdots + \eta_m|}{m}\, \mu(d\gamma) \\
&\le \lim_{k\to\infty} \sum_{l=1}^{k} \int_{\bar\Gamma_{\mathbb{R}^d}} \max_{2^{l-1}<m\le 2^l} \frac{|\eta_1 + \cdots + \eta_m|}{m}\, \mu(d\gamma) \\
&\le \sum_{l=1}^{\infty} \frac{1}{2^{(l-1)}} \int_{\bar\Gamma_{\mathbb{R}^d}} \max_{2^{l-1}<m\le 2^l} |\eta_1 + \cdots + \eta_m|\, \mu(d\gamma) \\
&\le \sum_{l=1}^{\infty} \frac{1}{2^{(l-1)}} E_\mu \left(\max_{m\le 2^l} (\eta_1 + \cdots + \eta_m)^2 \right)^{1/2} \\
&\le \sum_{l=1}^{\infty} \frac{1}{2^{(l-1)}} (c\, 2^l \log_2(2^{l+1})^2)^{1/2} \\
&\le c \sum_{l=1}^{\infty} (l+1)^2 2^{-l/2} < \infty.
\end{aligned}$$

Combining this estimate with Proposition 3, the proof is almost identical with that of Proposition 2 and therefore is omitted. □

Remark. Röckner and Schied [14] have also recently proved a capacitary version of the law of large numbers on configuration space using an interesting approach based on an intrinsic metric ρ. Indeed, using their metric it is easy to show $LLN^c \cap \Gamma_{\mathbb{R}^d} = \{\gamma \in \Gamma_{\mathbb{R}^d} : \rho(\gamma, \omega) < \infty$ for some $\omega \in LLN^c \cap \Gamma_{\mathbb{R}^d}\}$ so by [14, Corollary 3.2], $\mu(LLN^c) = 0$ implies that $LLN^c \cap \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$-exceptional. Their technique will also give a capacitary version of the law of the iterated logarithm.

However, the Dirichlet form used in [14] extends $(\mathcal{E}, D(\mathcal{E}))$, but it is not known whether the forms coincide. A form with a larger domain has more exceptional sets, so proofs of exceptionality are easier in Röckner and Schied's setting, and do not imply exceptionality in our setting.

5 Gibbs measures on configuration space

In this section we give some examples of Gibbs measures that satisfy our conditions for the law of large numbers, and the law of the iterated logarithm. Much of this section is adapted from [4] and [13].

Let σ be a measure on $\mathbb{R}^d$ that has a density ρ with respect to Lebesgue measure satisfying $\rho > 0$ almost everywhere, and $\rho^{1/2} \in H^{1,2}_{loc}(\mathbb{R}^d)$. Here $H^{1,2}_{loc}(\mathbb{R}^d)$ denotes the local Sobolev space of order 1 in $L^2_{loc}(\mathbb{R}^d; m)$. The *Poisson measure* π_σ with intensity measure σ is the probability measure on $\Gamma_{\mathbb{R}^d}$ characterized by:

$$\int_{\Gamma_{\mathbb{R}^d}} \exp(\langle f, \gamma \rangle)\, \pi_\sigma(d\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\, \sigma(dx) \right),$$

for $f \in C_0(\mathbb{R}^d)$. A *mixed Poisson measure* is given by:

$$\mu := \int_{\mathbb{R}_+} \pi_{z\sigma}\, \lambda(dz),$$

where λ is a probability measure on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} z\, \lambda(dz) < \infty$.

A *pair potential* is any measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x)$. For a pair potential ϕ, a bounded measurable subset Λ in $\mathbb{R}^d$, and a configuration $\gamma \in \Gamma_{\mathbb{R}^d}$, the *conditional energy* of γ in Λ is given by the formula

$$E^\phi_\Lambda(\gamma) := \begin{cases} \sum \phi(x-y) & \text{if } \sum |\phi(x-y)| < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

where the summation is taken over all pairs $\{x, y\} \subset \gamma$ such that $\{x, y\} \cap \Lambda \neq \emptyset$. We adopt the convention that a sum over the empty set is zero so that $E^\phi_\Lambda(\gamma) = 0$ if either $\gamma(\mathbb{R}^d) = 1$ or $\gamma(\Lambda) = 0$. We also define

$$Z^{z,\phi}_\Lambda(\gamma) := \int_{\Gamma_{\mathbb{R}^d}} \exp\left[-E^\phi_\Lambda(\gamma_{\Lambda^c} + \omega_\Lambda) \right] \pi_{z\sigma}(d\omega).$$

Here $\gamma_{\Lambda^c} + \omega_\Lambda$ is the configuration formed by combining the part of γ outside Λ with the part of ω inside Λ. The parameter $z > 0$ is called the *activity* .

Definition 3 *A probability measure μ on $\Gamma_{\mathbb{R}^d}$ is called a* Gibbs measure *with activity z, pair potential ϕ, and intensity measure σ if, for every* *bounded measurable* $\Lambda \subset \mathbb{R}^d$ *we have* $Z_\Lambda^{z,\phi}(\gamma) < \infty$ *for* μ*-almost every* $\gamma \in \Gamma_{\mathbb{R}^d}$ *and for every* $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$*,*

$$\mu(\Delta) = \int\limits_{\Gamma_{\mathbb{R}^d}}\int\limits_{\Gamma_{\mathbb{R}^d}} 1_\Delta(\gamma_{\Lambda^c} + \omega_\Lambda) \frac{\exp\left[-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda)\right]}{Z_\Lambda^{z,\phi}(\gamma)} \, \pi_{z\sigma}(d\omega)\, \mu(d\gamma). \qquad (7)$$

We recall the definition of the cubes $Q_r = r + (-1/2, 1/2]^d$ for $r \in \mathbb{Z}^d$.

Definition 4 *(SS) A pair potential* ϕ *is called* superstable *if there exist* $A > 0$ *and* $B \geq 0$ *so that if* $\Lambda = \cup_{r\in R} Q_r$ *is a finite union of cubes, then*

$$E_\Lambda^\phi(\gamma_\Lambda) \geq \sum_{r\in R} \left[A\gamma(Q_r)^2 - B\gamma(Q_r)\right].$$

(LR) A pair potential ϕ *is called* lower regular *if there exists a decreasing positive function* $\Psi : \mathbb{N} \to [0, \infty)$ *such that* $\sum_{r\in\mathbb{Z}^d} \Psi(|r|_\infty) < \infty$*, and for any disjoint* $\Lambda^{'}$ *and* $\Lambda^{''}$ *that are finite unions of cubes, then we have*

$$\int\limits_{\Lambda^{'}}\int\limits_{\Lambda^{''}} \phi(x - y)\, \gamma(dx)\gamma(dy) \geq - \sum_{r^{'}, r^{''} \in \mathbb{Z}^d} \Psi(|r^{'} - r^{''}|_\infty) \gamma_{\Lambda^{'}}(Q_{r^{'}}) \gamma_{\Lambda^{''}}(Q_{r^{''}}),$$

for all $\gamma \in \Gamma_{\mathbb{R}^d}$*. Here* $|\cdot|_\infty$ *refers to the maximum norm on* $\mathbb{R}^d$*.*

(I) A pair potential ϕ *is called* integrable *if* $\int_{\mathbb{R}^d} |\exp(-\phi(x)) - 1|\, dx < \infty$*.*

Definition 5 *A measure* μ *on* $\Gamma_{\mathbb{R}^d}$ *is called* tempered *if*

$$\limsup_{l\to\infty} \frac{\sum_{|r|\leq l} \gamma(Q_r)^2}{(2l+1)^d} < \infty \text{ for } \mu\text{-almost every } \gamma \in \Gamma_{\mathbb{R}^d}.$$

Definition 6 *A probability measure* μ *on* $\Gamma_{\mathbb{R}^d}$ *is called a* Ruelle measure *if* μ *is a tempered Gibbs measure with activity parameter* $z > 0$*, intensity* σ *equal to Lebesgue measure, and a pair potential* ϕ *that is superstable, lower regular, and integrable.*

Suppose that μ is a Gibbs measure and Λ a bounded measurable subset of $\mathbb{R}^d$. Let $\mathcal{F}(\Lambda)$ be the σ-algebra of events $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$ that only depend on the part of the configuration in Λ, that is, $1_\Delta(\gamma) = 1_\Delta(\gamma_\Lambda)$ for every

$\gamma \in \Gamma_{\mathbb{R}^d}$. Exchanging the order of integration in (7), we find that $\mu|_{\mathcal{F}(\Lambda)}$ is absolutely continuous with respect to $\pi_{z\sigma}|_{\mathcal{F}(\Lambda)}$ with density

$$\omega \mapsto \int_{\Gamma_{\mathbb{R}^d}} [Z_\Lambda^{z,\phi}(\gamma)]^{-1} \exp\left[-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda)\right] \mu(d\gamma). \tag{8}$$

In other words, a Gibbs measure is always locally absolutely continuous with respect to its corresponding Poisson measure. In general we have very little information about the density (8), but for Ruelle measures it is known to be bounded, with a bound that depends on $|\Lambda|$, the Lebesgue measure of Λ. In particular, for any $n \in \mathbb{N}$, the moments satisfy

$$E_\mu(\gamma(\Lambda)^n) \leq c(n, |\Lambda|), \tag{9}$$

and every Ruelle measure μ satisfies (μ.1).

Another important tool in studying Ruelle measures is the family of (infinite-volume) correlation functions $\rho_m : (\mathbb{R}^d)^m \to \mathbb{R}$ given by

$$\begin{aligned}&\rho_m(x_1, \ldots, x_m)\\ &\quad = z^m \exp(-\sum_{i<j} \phi(x_i - x_j)) \int_{\Gamma_{\mathbb{R}^d}} \exp\Big(-\sum_{i=1}^m \langle \phi(x_i - \cdot\,), \gamma\rangle\Big)\, \mu(d\gamma).\end{aligned}$$

These provide us with useful formulas for Ruelle measures:

$$\int_{\Gamma_{\mathbb{R}^d}} \langle f, \gamma\rangle\, \mu(d\gamma) = \int_{\mathbb{R}^d} f(x) \rho_1(x)\, dx, \tag{10}$$

$$\int_{\Gamma_{\mathbb{R}^d}} (\langle f, \gamma\rangle\langle g, \gamma\rangle - \langle fg, \gamma\rangle)\, \mu(d\gamma) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\, g(y) \rho_2(x, y)\, dx\, dy. \tag{11}$$

We now need additional smoothness and integrability assumptions on ϕ to ensure that μ satisfies (μ.2) and (μ.3) from Section 2. Sufficient conditions are obtained, for example, by adding to (SS) and (LR) the following:

(C) The function ϕ has compact support.

(D) The function $e^{-\beta\phi}$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d \backslash \{0\}$ and the weak gradient $\nabla\phi$, which is a locally m-integrable function on $\mathbb{R}^d \backslash \{0\}$, satisfies

$$\int_{\mathbb{R}^d} |\nabla\phi(x)| e^{-\beta\phi(x)}\, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla\phi(x)|^2 e^{-\beta\phi(x)}\, dx < \infty.$$

Condition (I) follows as a consequence, and the corresponding Ruelle measure μ satisfies (μ.1), (μ.2), and (μ.3). Thus [13, Theorem 4.8.2] ensures that the classical Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, given by closing the pre-Dirichlet form in Definition 2, is quasi-regular and local.

For sufficiently small z, by [17, Theorems 5.7 and 5.8], μ is the unique Gibbs measure for ϕ and z, and is translation invariant. In particular, the first correlation function $\rho_1(x) = \rho$ is constant.

Proposition 5 *Let μ be a Ruelle measure that satisfies (C) and (D). For sufficiently small z, conditions (μ.1)–(μ.5) hold so that the complement of the set*

$$\left\{\gamma \in \bar{\Gamma}_{\mathbb{R}^d} : \lim_{n\to\infty} \frac{\gamma(C_n)}{|C_n|} = \rho\right\}$$

is $\mathcal{E}$-exceptional.

Proof. Since μ is the unique Gibbs measure, it is translation-ergodic by [12, Theorem 4.1], so

$$E_\mu(\gamma(Q_0) \mid \mathcal{H}) = E_\mu(\gamma(Q_0)) = \rho, \ \mu\text{-a.e.}$$

which gives the result using Proposition 1 and Proposition 2. □

Suppose μ is a Ruelle measure whose pair potential ϕ satisfies conditions (C) and (D), specifically $\phi(x) = 0$ if $|x| \geq R$. In addition, suppose that the pair potential is non-negative, that is $\phi(x) \geq 0$ for all x. The non-negativity of ϕ allows us to say exactly how small z need be for our results to hold. Recall that we have assumed that the pair potential ϕ satisfies $C := \int [1 - \exp(-\phi(x))]\, dx < \infty$. Suppose that

$$z < (3eC)^{-1}. \tag{12}$$

This bound is more than sufficient, by [17, Theorems 5.7 and 5.8], to guarantee that μ is the unique Gibbs measure for ϕ and z, and is translation invariant. Now from (10) and (11) we get

$$\sigma_n^2 = \rho n + \int_{C_n}\int_{C_n} \omega_2(x,y)\, dx\, dy, \tag{13}$$

where $\omega_2 = \rho_2 - \rho^2$. For z satisfying (12), we have $\omega_2(x,y) = \omega_2(0, y-x)$.

Also, from (5.18) and (5.19) of [16, Section 4.5], we see that $z/(1+Cz) \leq \rho \leq z$, while (4.37) of [16, Section 4.4] (with $B=0$) gives

$$\int |\omega_2(0,x)|\, dx \leq z^2 eC/(1-zeC)^2.$$

Combining these inequalities we see that $\int |\omega_2(0,x)|\, dx < \rho$ if z satisfies (12). From (13) we conclude that there exist constants $a, b > 0$ so that

$$0 < a \leq \frac{\sigma_n^2}{|C_n|} \leq b < \infty. \tag{14}$$

In [22, Lemma 4], Spohn proved an exponential L^2-mixing for Gibbs fields at low density. He showed that if the potential is positive, and if z is sufficiently small (e.g. if the bound (12) holds), then condition (μ.7) holds. We simplified the expression from [22] by adjusting the constants c and α.

Proposition 6 *Let μ be a Ruelle measure that satisfies (C) and (D). Suppose that the potential is non-negative, and that (12) holds. Then conditions (μ.1)–(μ.7) hold so that $\mu(LIL^c) = 0$, and if $d \leq 2$ the set LIL^c is $\mathcal{E}$-exceptional.*

References

[1] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: Differential geometry of Poisson spaces, *C.R. Acad. Sci. Paris, t.* **323**, *Série I*, 1129–1134 (1996).

[2] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces, *C.R. Acad. Sci. Paris, t.* **323**, *Série I*, 1179–1184 (1996).

[3] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: Analysis and geometry on configuration spaces, *J. Funct. Anal.* **154**, 444–500 (1998).

[4] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: Analysis and geometry on configuration spaces: The Gibbsian case, *J. Funct. Anal.* **157**, 242–291 (1998).

[5] Yu. G. Kondratiev, R. A. Minlos, M. Röckner, and G. V. Shchepan'uk: Exponential mixing for classical continuous systems, In: *Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)*, 243–254, CMS Conf. Proc. 28, American Mathematical Society (2000).

[6] Z. M. Ma and M. Röckner: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Springer: Berlin, 1992.

[7] Z. M. Ma and M. Röckner: Construction of diffusions on configuration spaces, *Osaka J. Math.* **37**, 273–314, (2000).

[8] Z. M. Ma and M. Röckner: Diffusions on "simple" configuration spaces, In: *Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)*, 255–269, CMS Conf. Proc. 28, American Mathematical Society (2000).

[9] X. X. Nguyen and H. Zessin: Ergodic theorems for spatial processes, *Z. Wahr. verw. Gebiete* **48**, 133–158 (1979).

[10] H. Oodaira and K. Yoshihara: The law of the iterated logarithm for stationary processes satisfying mixing conditions, *Kodai Math. Sem. Rep.* **23**, 311–334 (1971).

[11] H. Osada: Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions, *Commun. Math. Phys* **176**, 117–131 (1996).

[12] C. Preston: *Random Fields*. Springer: Berlin, 1976.

[13] M. Röckner: Stochastic analysis on configuration spaces: Basic ideas and recent results, In: *New directions in Dirichlet forms.* AMS/IP Studies in Advanced Mathematics, Vol. 8, 157–231, American Mathematical Society (1998).

[14] M. Röckner and A. Schied: Rademacher's theorem on configuration spaces and applications, *J. Funct. Anal.* **169**, 325–356 (1999).

[15] M. Röckner and B. Schmuland: A support property for infinite-dimensional interacting diffusion processes, *C.R. Acad. Sci. Paris, t.* **326**, *Série I*, 359–364 (1998).

[16] D. Ruelle: *Statistical Mechanics: Rigorous Results*. W.A. Benjamin: New York, 1969.

[17] D. Ruelle: Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* **18**, 127–159 (1970).

[18] B. Schmuland: Some exceptional configurations, In: Stochastic Models ed. Luis Gorostiza and Gail Ivanoff, CMS Conference Proceedings, Volume 26, American Mathematical Society, 419–434 (2000).

[19] B. Schmuland: Extended Dirichlet spaces, *Comptes Rendus Mathematical Reports, Royal Society of Canada* **21**, 146–152 (1999).

[20] B. Schmuland and W. Sun: A central limit theorem and law of the iterated logarithm for a random field with exponential mixing, Preprint (2001). `www.stat.ualberta.ca/people/schmu/publications.html`

[21] R. J. Serfling: Moment inequalities for the maximum cumulative sum, *Ann. Math. Statist.* **41**, 1227–1234 (1970).

[22] H. Spohn: Equilibrium fluctuations for interacting Brownian particles, *Commun. Math. Phys.* **103**, 1–33 (1986).

[23] M. W. Yoshida: Construction of infinite dimensional interacting diffusion processes through Dirichlet forms, *Probab. Th. and Rel. Fields* **106**, 265–297 (1996).

[24] K. Yoshihara: The Borel-Cantelli lemma for strong mixing sequences of events and their applications to LIL, *Kodai Math. J.* **2**, 148–157 (1979).

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