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DIRICHLET FORMS WITH POLYNOMIAL DOMAIN

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ABSTRACT. We consider a pre-Dirichlet form $\mathcal{E}$ definedon a polynomial core. It is shown that the local and Markov properties for the closure $\overline{\mathcal{E}}$ follow from the product rule for the gradient; or more generally, from the product rule for the square of field operator. The usual functional calculus for the square of field operator is also established.

0. Introduction. When using Dirichlet forms to study diffusion processes one usually begins with a finite measure space $(E, \mathcal{B}, m)$ and a bilinear form of the type

$$\mathcal{E}(f,g) = 1/2 \int_E \langle \nabla f, \nabla g \rangle dm \tag{0.1}$$
$$\mathcal{D}(\mathcal{E}) = \mathcal{P}$$

where $\mathcal{P}$ is a suitable core of functions, dense in $L^2(m)$, and ∇ is somesort of gradient. Recall that such a non-negative definite, symmetric, bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$on $L^2(m)$ is *closed* if $\mathcal{D}(\mathcal{E})$ is complete with respect to the norm given by $\mathcal{E}_1 := \mathcal{E} + (\ ,\ )_{L^2(m)}$. The form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called *closable* if it has a closed extension, in which case its smallest closed extensionis called its *closure*. If $(\mathcal{E}, \mathcal{P})$ is closable, we then try to establish certain properties of its closure $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$. The Markov property, defined below, is especially important since it is a necessary and in many cases sufficient condition for the existence of an associated Markov process with state space E. The construction of this process can be found in [9] for the case where E is a locally compact separable metric space, and [1,2,4,10,13] for spaces E which are not locally compact.

Definition. Let $\mathcal{E}$ be a non-negative definite, symmetric, bilinear form on a (real) $L^2(E, \mathcal{B}, m)$ space, where m is a finite measure. The form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ has the Markov property if for any $\varepsilon > 0$ there exists $\phi_\varepsilon : \mathbb{R} \longrightarrow [-\varepsilon, 1+\varepsilon]$ satisfying:

(i) $\phi_\varepsilon(t) = t$ for all $t \in [0,1]$ and $0 \leq \phi_\varepsilon(t_2) - \phi_\varepsilon(t_1) \leq t_2 - t_1$ for all $t_1, t_2 \in \mathbb{R}, t_1 \leq t_2$.
(ii) $\phi_\varepsilon \circ u \in \mathcal{D}(\mathcal{E})$ for all$u \in \mathcal{D}(\mathcal{E})$.
(iii) $\mathcal{E}(\phi_\varepsilon \circ u, \phi_\varepsilon \circ u) \leq \mathcal{E}(u,u)$ for each $u \in \mathcal{D}(\mathcal{E})$.

Now if $\mathcal{P}$ is closed under composition with C_b^1 maps, and provided the gradient ∇ satisfies the chain rule, then the form given in (0.1) is clearly Markov. It is known that this entails the Markov property for the closure $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$. However, it is often convenient to begin with a core $\mathcal{P}$, of possibly unbounded functions, where $(\mathcal{E}, \mathcal{P})$ is not Markov. In this case, it can still happen that the closure $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ is Markov. In section 2, we show that it suffices to check the product rule for ∇ on the core $\mathcal{P}$ in order to guarantee the Markov property and the local property for the closed form $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$.

1. Approximation by polynomials over finite dimensional spaces. The first result we need gives a condition on μ so that the polynomials are dense in $L^2(\mathbb{R}^k;\mu)$. This is a known result which can be found in a paper by Dobrushin and Minlos [6], for example.

Lemma 1.1. *Let μ be a measure on $\mathbb{R}^k$ satisfying*

$$\int e^{2c|x|}\mu(dx) < \infty$$

for some $c > 0$, where $|x| = \sum_{j=1}^k |x_j|$. Then the polynomials are dense in $L^2(\mu)$.

Proof. If $g \in L^2(\mu)$, then the function $\phi : \mathbb{C}^k \to \mathbb{C}$ given by

$$\phi(z) = \phi(z_1, \dots, z_k) = \int e^{i\sum_{j=1}^k x_j z_j} g(x)\mu(dx)$$

is analytic in the region $\sup_{j=1}^{k} |Im(z_j)| < c$. If g is orthogonal to the subspace of polynomials, then ϕ vanishes together with all its derivatives at $z = 0$, and so $\phi \equiv 0$ throughout the region. In particular $\phi = 0$ on $\mathbb{R}^k$ so by the inversion theorem we find that

$$\int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} g(x)\mu(dx) = 0$$

for all but countably many $\{a_1, b_1, \dots, a_k, b_k\}$. Hence $g = 0$ in $L^2(\mu)$, which completes the proof. ∎

Remark. For a measure μ on the real line, a result of M. Riesz says that the polynomials are dense in $L^2(\mu)$ if and only if the measure $\nu(dx) = \mu(dx)/1+x^2$ is uniquely determined by its moments. We refer the interested reader to Freud's book [8] on orthogonal polynomials for details. In higher dimensions the situation is more complicated, and necessary and sufficient conditions do not appear to be known.

Definition. For $f \in C(\mathbb{R}^k)$ and $1 \le i \le k$ we define $I_i f \in C(\mathbb{R}^k)$ by

$$I_i f(x) = \int_0^{x_i} f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_k)dy$$

so that $\partial/\partial x_i(I_i f) = f$. By Cauchy-Schwarz it is easy to see that

$$(I_i f)^2(x) \le (I_i f^2)(x)x_i.$$

Definition. Suppose μ is a measure on $\mathbb{R}^k$ So that $\int e^{2c|x|}d\mu$ is finite for some $c > 0$. We define the measure $J_i\mu$ in the following way: First we disintegrate μ with respect to the projection $\pi_i : \mathbb{R}^k \to \{x_i = 0\} \approx \mathbb{R}^{k-1}$, that is, we find a kernel $\rho_i : \mathbb{R}^{k-1} \times \mathcal{B}(\mathbb{R}) \to [0, \infty)$ so that for all bounded, measurable u on $\mathbb{R}^k$ we have

$$\int_{\mathbb{R}^k} u(x)\mu(dx)$$
$$= \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_k)\rho_i(\pi_i(x); ds)d\pi_i(\mu).$$

Secondly, we define a new kernel

$$\tilde{\rho}_i(\pi_i(x); ds) = \begin{cases} (\int_s^\infty t\rho_i(dt))ds & s \geq 0 \\ (\int_{-\infty}^s -t\rho_i(dt))ds & s < 0 \end{cases}$$

and finally we let $J_i\mu$ be the measure with image $\pi_i\mu$ under π_i, and kernel $\tilde{\rho}_i$. We note that

$$\int e^{2c'|x|} dJ_i\mu < +\infty$$

for $c' < c$.

Lemma 1.2. *The map $I_i : L^2(J_i\mu) \to L^2(\mu)$ is continuous on the polynomials.*

Proof. $\int (I_iP)^2 d\mu \leq \int (I_iP^2)x_i d\mu = \int P^2 dJ_i\mu$, where the equality follows from Fubini's theorem.

Proposition 1.3. *Let μ satisfy $\int e^{2c|x|} d\mu < +\infty$ for some $c > 0$, and define*

$$||f||_1^2 = \int \nabla f \cdot \nabla f d\mu + \int f^2 d\mu.$$

If $f \in C^1(\mathbb{R}^k)$, and $||f||_1 < \infty$, then f can be approximated in $||\cdot||_1$-norm by polynomials.

Proof. If $||f||_1 < +\infty$, then by multiplying with appropriate bump functions it is easy to show that f can be approximated in $||\cdot||_1$-norm by functions with compact support. Accordingly, we take f with compact support and without loss of generality, assume the support lies in $\{x_1 > 0, \ldots, x_k > 0\}$. Also the convolution of f with smooth mollifiers whose compact supports shrink to zero, yields a sequence $\{f_n\}$ of C_0^∞ functions so $f_n \to f$ and $\nabla f_n \to \nabla f$ uniformly. So without loss of generality we assume that $f \in C_0^\infty(\mathbb{R}^k)$. Define the measure $\nu = \sum_{i=1}^k J_k(\cdots J_{i+1}(J_{i-1}(\cdots(J_1\mu)\cdots))) + J_k(\cdots(J_1\mu)\cdots)$. By Lemma 1.1 we can choose a sequence $\{P_n\}$ of polynomials so

$$P_n \longrightarrow \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_k} f$$

in $L^2(\nu)$. If we define the polynomials $Q_n = I_1 \ldots I_k P_n$, then repeated use of Lemma 1.2. shows that

$$\frac{\partial}{\partial x_i} Q_n \to \frac{\partial}{\partial x_i} f \quad \forall i = 1, \ldots, k$$

and $Q_n \to f$, all convergence taking place in $L^2(\mu)$.

2. The product rule and some consequences. Let $(E, \mathcal{B}, m)$ be a finite measure space and suppose $\mathcal{U}$ is a linear subspace of $L^2(m)$ satisfying

(i) For each $u \in \mathcal{U}$, there exist constants $K, c > 0$ so that for all $t \geq 0$ we have

$$m\{z : |u(z)| > t\} \leq Ke^{-ct}. \tag{2.1}$$

(ii) The σ-algebra generated by $\mathcal{U}$ is $\mathcal{B}$.

These two assumptions, combined with Lemma 1.1, show that the algebra generated by $\mathcal{U} \cup \{1\}$ is dense in L^2. That is, if we let

$$\mathcal{P} = \{P(u_1, \ldots, u_k) : \ u_i \in \mathcal{U} \quad 1 \leq i \leq k, \ P \text{ is apolynomial on } \mathbb{R}^k\},$$

then $\mathcal{P} \subset L^2$ and is dense in the L^2-metric. Suppose $\Gamma : \mathcal{U} \times \mathcal{U} \to L^\infty(m)$ is a symmetric bilinear map such that $\Gamma(u,u) \geq 0$ a.e. for all $u \in \mathcal{U}$. We also assume that the product rule holds on $\mathcal{U}$, i.e., If $f, g, h, fg \in \mathcal{U}$, then

$$\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h) \quad \text{a.e.} \tag{2.2}$$

Using linearity, the product rule and setting $\Gamma(1,u) = 0$ for all $u \in \mathcal{U}$, we find that Γ can be extended to $\mathcal{P}$ as a mapping into $L^1(m)$. We now set

$$\begin{aligned} \mathcal{E}(f,g) &= 1/2 \int \Gamma(f,g)dm \\ \mathcal{D}(\mathcal{E}) &= \mathcal{P} \end{aligned} \tag{2.3}$$

which is a densely defined, non-negative definite, symmetric bilinear form on $L^2(m)$. We will assume that $(\mathcal{E}, \mathcal{P})$ is closable and let $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ denote its closure.

Remark. The map Γ is called the square of field operator and the equation (2.5) in the statement of Proposition 2.2 is called its functional calculus. Bouleau and Hirsch [5] established this result for a closed form which was assumed to be Markov, local and satisfy the hypothesis of representability:

$$2\mathcal{E}(fh,f) - \mathcal{E}(h,f^2) = \int h\Gamma(f,f)dm$$

for all $f, h \in \mathcal{D}(\mathcal{E}) \cap L^\infty$. Note that the product rule (2.2) gives us this equation for the form $(\mathcal{E}, \mathcal{P})$ in (2.3). Our approach is the reverse of that used by Bouleau and Hirsch; we assume (2.2) and (2.3) hold on a core $\mathcal{P}$, and first establish the functional calculus for the square of field operator. Then the Markov and local properties for the closure follow as corollaries.

Lemma 2.1. *If $\{p_n\}$ and $\{q_n\}$ are $\mathcal{E}$-Cauchy sequences in $\mathcal{P}$, then $\Gamma(p_n, q_n)$ is Cauchy in $L^1(m)$.*

Proof. The positivity of Γ gives us a Cauchy-Schwarz inequality

$$|\Gamma(f,g)| \leq \Gamma^{1/2}(f,f)\Gamma^{1/2}(g,g) \quad a.e. \tag{2.4}$$

Thus we see for $p_n,\ p_m,\ q_n,\ q_m \in \mathcal{P}$

$$\begin{aligned} &|\Gamma(p_n,q_n) - \Gamma(p_m,q_m)| \\ &\quad \leq |\Gamma(p_n, q_n - q_m)| + |\Gamma(p_n - p_m, q_m)| \\ &\quad \leq \Gamma^{1/2}(p_n,p_n)\Gamma^{1/2}(q_n - q_m, q_n - q_m) \\ &\qquad + \Gamma^{1/2}(p_n - p_m, p_n - p_m)\Gamma^{1/2}(q_m,q_m). \end{aligned}$$

Integrating with respect to m, and using Cauchy-Schwarz in $L^2(m)$ yields

$$\begin{aligned} &1/2 \int |\Gamma(p_n,q_n) - \Gamma(p_m,q_m)| \\ &\qquad \leq \mathcal{E}^{1/2}(p_n,p_n)\mathcal{E}^{1/2}(q_n - q_m, q_n - q_m) \\ &\qquad\quad + \mathcal{E}^{1/2}(q_m,q_m)\mathcal{E}^{1/2}(p_n - p_m, p_n - p_m), \end{aligned}$$

so that if $\{p_n\}$ and $\{q_n\}$ are $\mathcal{E}$-Cauchy then $\Gamma(p_n, q_n)$ is Cauchy in $L^1(m)$. ∎

Using the previous lemma we find that Γ can be extended to $\overline{\mathcal{P}}$ in a unique way so

(i) $\Gamma(f,f) \in L^1(m)$ and $\Gamma(f,f) \geq 0$ a.e. for $f \in \overline{\mathcal{P}}$.

(ii) $\overline{\mathcal{E}}(f,g) = 1/2 \int \Gamma(f,g)dm$ for $f, g \in \overline{\mathcal{P}}$.

(iii) $f_n \to f, g_n \to g$ in $\mathcal{E}_1$-norm implies $\Gamma(f_n, g_n) \to \Gamma(f,g)$ in $L^1(m)$.

Definition. If $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_k) \in (\overline{\mathcal{P}})^k$ we let $\Gamma(f,g)$ be the $k \times k$ – matrix valued function

$$(\Gamma(f,g))_{ij} = \Gamma(f_i, g_j).$$

Notice that $\Gamma(f,f)$ is (a.e.) symmetric, and positive definite. For ease of notation we write $\Gamma(f) = \Gamma(f,f)$.

From the product rule (2.2) it follows that if $u = (u_1, \ldots, u_n) \in \mathcal{U}^n$ and P, Q are polynomials on $\mathbb{R}^n$, then

$$\Gamma(P(u), Q(u)) = \langle \nabla P(u), \Gamma(u) \nabla Q(u) \rangle.$$

Hence if $p = P(u) = (P_1(u), \ldots, P_k(u))$ where $\{P_i\}$are polynomials on $\mathbb{R}^n$, then

$$(\Gamma(p))_{ij} = \langle \nabla P_i(u), \Gamma(u) \nabla P_j(u) \rangle.$$

The chain rule now shows us that for $\phi \in C^1(\mathbb{R}^k)$, we have

$$\langle \nabla(\phi \circ P)(u), \Gamma(u) \nabla(\phi \circ P)(u) \rangle = \langle (\nabla\phi)(p), \Gamma(p)(\nabla\phi)(p) \rangle.$$

This equation is a crude version of the chain rule for $\overline{\mathcal{P}}$ that we will demonstrate below.

Proposition 2.2. *If $f = (f_1, \ldots, f_k) \in (\overline{\mathcal{P}})^k$ and $\phi \in C_b^1(\mathbb{R}^k)$, then $\phi(f) \in \overline{\mathcal{P}}$ and*

$$\Gamma(\phi(f)) = \langle (\nabla\phi)(f), \Gamma(f)(\nabla\phi)(f) \rangle. \tag{2.5}$$

Proof. First we prove the result for functions f of the form

$$p = P(u) = (P_1(u), \ldots, P_k(u)) \in \mathcal{P}^k$$

where $u = (u_1, \ldots, u_n) \in \mathcal{U}^n$. Let μ be the distribution of u on $\mathbb{R}^n$. By assumption (2.1) and Lemma 1.1 we can find polynomials Q_n converging to $\phi \circ P$ in $\|\cdot\|_1$-norm. By taking subsequences we can assume that $Q_n \to \phi \circ P$ and $\nabla Q_n \to \nabla(\phi \circ P)$ μ-a.e. on $\mathbb{R}^n$, which means $Q_n(u) \to \phi \circ P(u)$ and $(\nabla Q_n)(u) \to \nabla(\phi \circ P)(u)$ m-a.e. on E. Now since $\Gamma(u) \in L^\infty(m)$ componentwise, there is a constant $c > 0$ so

$$\begin{aligned}
&\int \langle (\nabla Q_n(u) - \nabla Q_m(u)), \Gamma(u)(\nabla Q_n(u) - \nabla Q_m(u)) \rangle dm \\
&\qquad \leq c \int_E \|\nabla Q_n(u) - \nabla Q_m(u)\|^2 dm \\
&\qquad = c \int_{\mathbb{R}^n} \|\nabla Q_n - \nabla Q_m\|^2(x) \mu(dx)
\end{aligned}$$

which shows that $Q_n(u)$ is $\mathcal{E}_1$-Cauchy. By the pointwise convergence we know the limit is $(\phi \circ P)(u)$ and so $\phi \circ P(u) \in \overline{\mathcal{P}}$ with

$$\begin{aligned}\Gamma(\phi \circ P(u)) &= \lim_n \Gamma(Q_n(u))\\ &= \lim_n \langle (\nabla Q_n)(u), \Gamma(u)(\nabla Q_n)(u)\rangle\\ &= \langle \nabla(\phi \circ P)(u), \Gamma(u)\nabla(\phi \circ P)(u)\rangle\\ &= \langle (\nabla\phi)(p), \Gamma(p)(\nabla\phi)(p)\rangle.\end{aligned}$$

We've shown that the result is true for $p \in \mathcal{P}^k$, we now want prove it for $f \in (\overline{\mathcal{P}})^k$. Let us fix $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots q_k) \in \mathcal{P}^k, \phi \in C_b^1(\mathbb{R}^k)$ and let $c = \sup\{|\partial\phi/\partial x_i(x)| : x \in \mathbb{R}^k, 1 \leq i \leq k\}$. We've just seen that $\phi \circ p$ and $\phi \circ q$ belong to $\overline{\mathcal{P}}$, we'd also like to show that they are close together if p and q are. Consider the equation

$$\begin{aligned}\Gamma(\phi \circ p) - &\Gamma(\phi \circ p, \phi \circ q)\\ &= \langle (\nabla\phi)(p), \Gamma(p)(\nabla\phi)(p)\rangle - \langle (\nabla\phi)(p), \Gamma(p,q)(\nabla\phi)(q)\rangle\\ &= \langle (\nabla\phi)(p), \Gamma(p)[(\nabla\phi)(p) - (\nabla\phi)(q)]\rangle\\ &\quad + \langle (\nabla\phi)(p), \Gamma(p, p-q)(\nabla\phi)(q)\rangle.\end{aligned} \tag{2.6}$$

Let's first look at the second summand on the right hand side. Integrating with respect to dm, and using (2.4) combined with Cauchy-Schwarz on $L^2(m)$ we obtain the bound

$$\begin{aligned}\int |\langle (\nabla\phi)(p), &\Gamma(p, p-q)(\nabla\phi)(q)\rangle| dm\\ &\leq c^2 \sum_{i,j=1}^k \int |\Gamma(p_i, p_j - q_j)| dm\\ &\leq c^2 \sum_{i,j=1}^k \int \Gamma^{1/2}(p_i)\Gamma^{1/2}(p_j - q_j) dm\\ &\leq c^2 \sum_{i,j=1}^k \Big(\int \Gamma(p_i) dm\Big)^{1/2} \Big(\int \Gamma(p_j - q_j) dm\Big)^{1/2}\\ &= 2c^2 \Big(\sum_{i=1}^k \mathcal{E}^{1/2}(p_i, p_i)\Big)\Big(\sum_{j=1}^k \mathcal{E}^{1/2}(p_j - q_j, p_j - q_j)\Big).\end{aligned} \tag{2.7}$$

We are now in a position to prove the main result. Suppose $f \in (\overline{\mathcal{P}})^k$ andlet $p_n = (p_{1,n}, \ldots, p_{k,n}) \in \mathcal{P}^k$ converge to f in $\mathcal{E}_1$-norm componentwise. By taking a subsequence we may also assume that $p_{i,n} \to f_i$ and $\Gamma(p_{i,n}) \to \Gamma(f_i)$ almost everywhere on E, for $i = 1, \ldots, k$. Now $\Gamma(\phi \circ p_n - \phi \circ p_m) = \Gamma(\phi \circ p_n) + \Gamma(\phi \circ p_m) - 2\Gamma(\phi \circ p_n, \phi \circ p_m)$ so using

(2.6) and (2.7) we obtain

$$\begin{aligned}
&\int \Gamma(\phi\circ p_n - \phi\circ p_m)dm \\
&\quad\le \int |\langle(\nabla\phi)(p_n), \Gamma(p_n)[(\nabla\phi)(p_n) - (\nabla\phi)(p_m)]\rangle| dm \\
&\quad + 2c^2\Big(\sum_{i=1}^{k}\mathcal{E}^{1/2}(p_{i,n},p_{i,n})\Big)\Big(\sum_{j=1}^{k}\mathcal{E}^{1/2}(p_{j,n}-p_{j,m},p_{j,n}-p_{j,m})\Big) \\
&\quad + \int |\langle(\nabla\phi)(p_m), \Gamma(p_m)[(\nabla\phi)(p_n) - (\nabla\phi)(p_m)]\rangle| dm \\
&\quad + 2c^2\Big(\sum_{i=1}^{k}\mathcal{E}^{1/2}(p_{i,m},p_{i,m})\Big)\Big(\sum_{j=1}^{k}\mathcal{E}^{1/2}(p_{j,n}-p_{j,m},p_{j,n}-p_{j,m})\Big).
\end{aligned}$$

This estimate, combined with the fact that p_n is $\mathcal{E}$-Cauchy, that $\Gamma(p_n)$ converges to $\Gamma(f)$ in L^1 componentwise, and that $(\nabla\phi)(p_n)$ converges to $(\nabla\phi)(f)$ boundedly pointwise tells us that $\phi\circ p_n$ is $\mathcal{E}$-Cauchy. Its pointwise limit is $\phi\circ f$ so we conclude that $\phi\circ f \in \overline{\mathcal{P}}$ and

$$\begin{aligned}
\Gamma(\phi\circ f) &= \lim_n \Gamma(\phi\circ p_n) \\
&= \lim_n \langle(\nabla\phi)(p_n), \Gamma(p_n)(\nabla\phi)(p_n)\rangle \\
&= \langle(\nabla\phi)(f), \Gamma(f)(\nabla\phi)(f)\rangle.
\end{aligned}$$

Corollary 2.3. *$(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ has the Markov property.*

Proof. Let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be $C_b^\infty(\mathbb{R})$ functions satisfying (i) in the definition of the Markov property. If $f\in\overline{\mathcal{P}}$, then Proposition 2.2 tells us that, for every $\varepsilon>0$, $\phi_\varepsilon\circ f\in\overline{\mathcal{P}}$ and

$$\Gamma(\phi_\varepsilon\circ f) = (\phi_\varepsilon'(f))^2\Gamma(f).$$

Since $\phi_\varepsilon'(x)\le 1$ for all $x\in\mathbb{R}$, we conclude that

$$\begin{aligned}
\overline{\mathcal{E}}(\phi_\varepsilon\circ f,\phi_\varepsilon\circ f) &= \frac{1}{2}\int\Gamma(\phi_\varepsilon\circ f)dm \\
&= \frac{1}{2}\int(\phi_\varepsilon'(f))^2\Gamma(f)dm \\
&\le \frac{1}{2}\int\Gamma(f)dm \\
&= \overline{\mathcal{E}}(f,f),
\end{aligned}$$

which gives us the Markov property. ∎

Corollary 2.4. *The form $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ has the local property, i.e., if u, v belong to $\overline{\mathcal{P}}$ and $uv = 0$ m–a.e., then $\Gamma(u,v) = 0$ m–a.e. (and thus $\mathcal{E}(u,v)=0$).*

Proof. Choose a function $\phi\in C_b^1(\mathbb{R})$ such that $\phi(0)=0$, and $\phi'(x)>0$ for all $x\in\mathbb{R}$. Then $\phi(u)\phi(v)=0$ a.e., and $\phi(u),\phi(v)\in\overline{\mathcal{P}}$ with

$$\Gamma(\phi(u),\phi(v)) = \phi'(u)\phi'(v)\Gamma(u,v).$$

This shows that $\Gamma(\phi(u),\phi(v))=0$ a.e. if and only if $\Gamma(u,v)=0$ a.e., so it suffices to prove the result for $\phi(u)$ and $\phi(v)$. Consequently we assume, without loss of generality, that u and v are bounded. If $u,v\in\overline{\mathcal{P}}$ and are bounded, then $uv\in\overline{\mathcal{P}}$ and for every $h\in\overline{\mathcal{P}}$,

$$\Gamma(uv,h)=u\Gamma(v,h)+v\Gamma(u,h).$$

However $uv=0$ so $\Gamma(uv,h)=0$ a.e. which gives us

$$u\Gamma(v,h)+v\Gamma(u,h)=0 \quad m-\text{ a.e.} \tag{2.8}$$

Setting $h=u$ and multiplying (2.8) by u yields

$$u^2\Gamma(u,v)=0 \quad m-\text{ a.e.}$$

This shows that $\Gamma(u,v)=0$, at least on the set where $u\neq 0$. We must work a bit more to show that $\Gamma(u,v)=0$ a.e. on the set $\{x: u(x)=0\}$. For $\varphi,\psi\in C_0^\infty(\mathbb{R})$ we have $\varphi(u),\psi(u)\in\overline{\mathcal{P}}$ with

$$\begin{aligned}\overline{\mathcal{E}}(\varphi(u),\psi(u))&=\frac{1}{2}\int\varphi'(u)\psi'(u)\Gamma(u)dm\\ &=\frac{1}{2}\int_{\mathbb{R}}\varphi'(x)\psi'(x)\mu(dx)\end{aligned} \tag{2.9}$$

and

$$\int\varphi(u)\psi(u)dm=\int\varphi(x)\psi(x)\nu(dx),$$

where μ is the distribution of u under $\Gamma(u)dm$, and ν is the distribution of u under dm. Since $\overline{\mathcal{E}}$ is a closed form on $L^2(E;m)$ we see that (2.9) defines a closable form on $C_0^\infty(\mathbb{R})\subset L^2(\mathbb{R};\nu)$. By Theorem 2.2 of [3], we conclude that μ must be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Thus

$$0=\mu(\{0\})=\int_{\{u=0\}}\Gamma(u)dm,$$

so that $\Gamma(u)=0$ $m-$ a.e. on $\{u=0\}$. Since

$$|\Gamma(u,v)|\leq\Gamma^{1/2}(u)\Gamma^{1/2}(v),$$

we see that $\Gamma(u,v)=0$ $m-$a.e. on $\{u=0\}$. This shows that $\Gamma(u,v)=0$ almost everywhere and concludes the proof. ∎

3. Examples. 1. The Ornstein-Uhlenbeck process. Let H be a real separable Hilbert space and A a self-adjoint operator on H satisfying $A\geq cI$, for some $c>0$. We suppose H is densely and continuously embedded into a Banach space E large enough to support the Gaussian measure with covariance A^{-1}. That is, we suppose there is a measure m on the Borel sets of E, so that the collection of random variables

$$\mathcal{U}=\{\langle\ell,\cdot\rangle:\ \ell\in E^*\},$$

is a mean zero Gaussian family with covariance

$$\int_E\langle\ell,\cdot\rangle\langle k,\cdot\rangle dm=(\ell,A^{-1}k).$$

Here the pointed brackets $\langle\cdot,\cdot\rangle$ refer to the pairing between E and E^*, while the round brackets $(\cdot,\cdot)$ denote the inner product on H. Also, we are using the injection dual to $H \to E$ to map E^* into $H^* \simeq H$. Since m is Gaussian, condition (2.1) holds and $\mathcal{U}$ is easily seen to generate the Borel sets in E. For $u = \langle \ell, \cdot\rangle$ and $v = \langle k, \cdot\rangle$, we set $\Gamma(u,v) = (\ell, k)$ so that $\Gamma(u,u) \in L^\infty$, and the product rule holds vacuously. The form defined in (2.3) is closable, so Proposition 2.2 therefore applies, and as a result $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ is local and Markov. The corresponding Markov process is the Ornstein-Uhlenbeck process in E, with drift operator A and driven by white noise W on H. It is a weak solution to the stochastic evolution equation

$$dX = -AX + W.$$

The results in [13] show that X has strongly continuous sample paths in E.

2. Energy forms on polynomial domains in the context of quantum field theory were considered in a paper by Potthoff and Röckner [12]. In their set up the state space E was $\mathcal{S}'(\mathbb{R}^d)$ for some $d \geq 1$, and ν was a measure on E given by a positive white noise functional, with supp $\nu = E$. Their core, assumed dense in $L^2(\nu)$, was given by

$$\mathcal{P} = \{P(\langle \xi_1, \cdot\rangle, \ldots, \langle \xi_m, \cdot\rangle) : \ m \in \mathbb{N}, \ \xi_1, \ldots, \xi_m \in \mathcal{S}(\mathbb{R}^d) \text{ and } P \text{ is a real polynomial on } \mathbb{R}^m\}.$$

Potthoff and Röckner defined a form by

$$\mathcal{E}(u,v) = \int_{S'} (\nabla u(z), \nabla v(z))_{L^2} \nu(dz)$$
$$\mathcal{D}(\mathcal{E}) = \mathcal{P},$$

where for $z \in \mathcal{S}'(\mathbb{R}^d)$ and $u \in \mathcal{P}$, $\nabla u(z)$ is the unique element in $L^2(\mathbb{R}^d)$ representing the continuous linear functional

$$h \to \frac{\partial u}{\partial h}(z) := \lim_{s\to 0} \frac{u(z+sh) - u(z)}{s}, \quad h \in L^2(\mathbb{R}^d).$$

They also assumed that $(\mathcal{E}, \mathcal{P})$ was closable and showed in Theorem 1.3 of [10] that $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ was Markov. This result can also be recovered from our Proposition 2.2 by setting

$$\mathcal{U} = \{\langle \ell, \cdot\rangle : \ \ell \in \mathcal{S}(\mathbb{R}^d)\}$$

and

$$\Gamma(\langle \ell, \cdot\rangle, \langle k, \cdot\rangle) = (\ell, k)_{L^2} \in L^\infty(\nu).$$

Recent results by Lee [11] show that since ν comes from a positive white noise functional, there exists a Hilbert space $\mathcal{S}_{-p}$ which supports ν and which satisfies

$$\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{S}_p(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \subseteq \mathcal{S}_{-p}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$$

and

$$\int_{\mathcal{S}_{-p}} \exp(\frac{1}{2}\|z\|_{-p}^2)\nu(dz) < \infty.$$

Here $\mathcal{S}_p$ is the dual of $\mathcal{S}_{-p}$, so for any $\ell \in \mathcal{S}$ we have $|\langle \ell, z\rangle| \leq \|\ell\|_p \|z\|_{-p}$. Combined with the exponential moment above, this gives us condition (2.1) for $\mathcal{U}$ and so our results applyhere.

3. The energy form associated with the infinitely many neutral all eles diffusion model was introduced in [14] on a polynomial domain. In this case, the state space E is compact and the members of $\mathcal{U}$ continuous, hence bounded, so (2.1) holds trivially. In [14], the Markov property of $(\overline{\mathcal{E}}, \overline{\mathcal{P}})$ is proven directly, but it also follows from Corollary 2.3. Also, combining Corollary 2.4 with Theorem 4.5.3 of [9] gives a new proof of the sample path continuity of the process, which was originally proved by Ethier and Kurtz [7].

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