

# Reversibility of Interacting Fleming-Viot Processes with Mutation, Selection, and Recombination \*

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## Abstract

Reversibility of the Fleming-Viot process with mutation, selection, and recombination is well understood. In this paper, we study the reversibility of a system of Fleming-Viot processes that live on a countable number of colonies interacting with each other through migrations between the colonies. It is shown that reversibility fails when both migration and mutation are non-trivial.

## 1 Introduction

The Fleming-Viot process is a probability-measure-valued Markov process describing the evolution of the distribution of allelic types in a large population. It arises most naturally in population genetics as the limit in distribution of certain sequences of Markov chains undergoing mutation, natural selection, recombination, and random genetic drift.

Reversibility plays an important role in statistical inference in the neutral theory of population genetics. When reversibility holds, techniques used for future predictions can then be used to understand the starting distribution that lead to the present state. Several models, such as the Wright-Fisher Markov chain and the finite alleles Wright-Fisher diffusion, are reversible. The reversibility of the Fleming-Viot process with parent independent mutation was obtained in [3] and [12]. On the other hand, reversibility is a very restrictive property. The results in [10], [9], and [8], show that the

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Fleming-Viot process is reversible only if the mutation, natural selection, and recombination have a special form.

The interacting Fleming-Viot process studied in this paper is a countable collection of Fleming-Viot processes that interact through geographical migration. It is the diffusion approximation to the stepping-stone model involving infinitely many alleles. Without migration, our system will simply be a collection of independent Fleming-Viot processes. The migration can be viewed as an external force acting upon the independent system of the Fleming-Viot processes. Since the internal reversible forces such as mutation and selection are constantly corrected by the external migration force, it is natural to expect the loss of reversibility in the interacting Fleming-Viot process due to competition between local forces and migration.

The long-time behavior of the interacting Fleming-Viot process is well-known. In the absence of mutation, selection, and recombination, a complete characterization of stationary distributions were obtained in [11] for the two allele case and in [1] for the general case in terms of migration. In [13] (two allele) and [2] (general), the structures of the stationary distributions were investigated for models involving mutation, selection and recombination. In this paper we study the reversibility of the general Fleming-Viot process and investigate the interrelation between mutation, selection and recombination, and migration. Under very general hypotheses, we show that the interacting Fleming-Viot process with mutation, selection, recombination, and migration is irreversible. Our results cover all models in [11], [13], [1], and [2].

## 2 Model

Let  $I$  be a countable index set where each element  $\xi \in I$  labels a colony. The different genetic types of individuals in the population will be modelled by a compact metric space  $E$ . Let  $M_1(E)$  denote the space of Borel probability measures on  $E$ ,  $M(E)$  be the space of finite signed Borel measures on  $E$ , and  $\Delta$  the set of Dirac measures on  $E$ . We let  $B(E)$  denote the space of bounded measurable functions on  $E$ , and  $C(E)$  the space of continuous functions on  $E$ . For any  $\mu$  in  $M(E)$  and  $g$  in  $B(E)$ , we use the notation  $\langle \mu, g \rangle = \int_E g(x) \mu(dx)$ . Let

$$\begin{aligned} B(E)^I &:= \{\mathbf{f} = (f_\xi)_{\xi \in I} : f_\xi \in B(E)\} \\ M(E)^I &:= \{X = (X_\xi)_{\xi \in I} : X_\xi \in M(E)\}. \end{aligned}$$

For  $X$  in  $M(E)^I$  and  $\mathbf{f}$  in  $B(E)^I$ , we write  $\langle X, \mathbf{f} \rangle := \sum_{\xi \in I} \langle X_\xi, f_\xi \rangle$  whenever the sum converges. The state space for our process will be  $M_1(E)^I \subseteq M(E)^I$ .

For every  $\xi, \xi'$  in  $I$ , let  $a(\xi, \xi')$  denote the migration probability from colony  $\xi$  to colony  $\xi'$ . We assume

$$a(\xi, \xi) = 0, \quad \sum_{\xi' \in I} a(\xi, \xi') = 1. \quad (2.1)$$

Define the mutation operator  $(A, \mathcal{D}(A))$  to be the generator of a conservative Feller semigroup  $(P_t)$  on  $C(E)$ . We assume that the domain  $\mathcal{D}(A)$  of  $A$  is dense in  $C(E)$ .

The sets  $C(E)^I$  and  $\mathcal{D}(A)^I$  denote subsets of  $B(E)^I$ , where the coordinate functions are in  $C(E)$  and  $\mathcal{D}(A)$ , respectively. Set

$$B(E)_0^I := \{\mathbf{f} \in B(E)^I : f_\xi \equiv 0 \text{ for all } \xi \text{ outside a finite subset of } I\},$$

and define  $C(E)_0^I$  and  $\mathcal{D}(A)_0^I$  similarly.

For any symmetric bounded measurable function  $V$  on  $E^2$ , we define the selection operator  $S : M_1(E) \rightarrow M(E)$  by

$$S(\mu)(du) := \left( \int_E V(u, v) \mu(dv) - \int_E \int_E V(v, w) \mu(dv) \mu(dw) \right) \mu(du).$$

When two types  $u, v$  undergo recombination; the distribution of the resulting type is distributed according to the probability kernel  $\eta(u, v; dw)$ . The recombination operator  $R : M_1(E) \rightarrow M(E)$  is given by

$$R(\mu)(du) := \int_E \int_E \eta(v, w; du) \mu(dv) \mu(dw) - \mu(du).$$

Let  $\tilde{\mathcal{A}}$  be the algebra of functions on  $M_1(E)^I$  given by the collection of linear combinations of functions of the form

$$F(X) = \prod_{i=1}^m \langle X_{\xi_i}, f_i \rangle, \quad (2.2)$$

where  $m \geq 1$ ,  $f_i \in B(E)$  for  $1 \leq i \leq m$ , and  $(\xi_1, \dots, \xi_m) \in I^m$ . Similarly, let  $\mathcal{A}$  be the sub-algebra of  $\tilde{\mathcal{A}}$ , given by linear combinations of functions of the form (2.2) with  $f_i \in \mathcal{D}(A)$  for  $1 \leq i \leq m$ . Note that both  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$  are measure determining on  $M_1(E)^I$ .

For  $F : M_1(E)^I \rightarrow \mathbb{R}$  we define partial derivatives as follows, whenever the limit exists:

$$\frac{\partial F(X)}{\partial X_\xi}(u) := \lim_{\varepsilon \downarrow 0} \frac{F(X^\varepsilon(\xi, u)) - F(X)}{\varepsilon} \quad \text{for } u \in E, \xi \in I,$$

with

$$(X^\varepsilon(\xi, u))_{\xi'} := \begin{cases} X_{\xi'} & \text{if } \xi' \neq \xi, \\ X_\xi + \varepsilon \delta_u & \text{if } \xi' = \xi. \end{cases}$$

This definition requires us to extend the domain of  $F$  infinitesimally from  $M_1(E)^I$  to  $M(E)^I$ . For  $F$  in  $\tilde{\mathcal{A}}$ , this is done via (2.2).

For non-negative numbers  $s, r, \rho$ , the generator  $\mathcal{L}_{s,r,\rho}$  of the interacting Fleming-Viot process incorporating migration, mutation, selection, and recombination is defined for  $F \in \mathcal{A}$  by

$$\mathcal{L}_{s,r,\rho} F(X) := \mathcal{L}_{s,r} F(X) + \mathcal{L}_\rho F(X),$$

where

$$\mathcal{L}_\rho F(X) := \rho \sum_{\xi, \xi' \in I} a(\xi, \xi') \left\langle X_{\xi'} - X_\xi, \frac{\delta F}{\delta X_\xi(\cdot)} \right\rangle, \quad (2.3)$$

$$\begin{aligned} \mathcal{L}_{s,r} F(X) &:= \sum_{\xi \in I} \left\langle X_\xi, A \frac{\delta F}{\delta X_\xi(\cdot)} \right\rangle + s \sum_{\xi \in I} \left\langle S(X_\xi), \frac{\delta F}{\delta X_\xi(\cdot)} \right\rangle + r \sum_{\xi \in I} \left\langle R(X_\xi), \frac{\delta F}{\delta X_\xi(\cdot)} \right\rangle \\ &+ \frac{1}{2} \sum_{\xi \in I} \int_E \int_E \frac{\delta^2 F}{\delta X_\xi(u) \delta X_\xi(v)} Q_{X_\xi}(du, dv), \end{aligned}$$

and

$$Q_\mu(du, dv) := \mu(du)\delta_u(dv) - \mu(du)\mu(dv).$$

For  $X \in M_1(E)^I$  and  $\mathbf{f} \in \mathcal{D}(A)_0^I$ , define

$$\langle b_\xi(X), f_\xi \rangle := \langle X_\xi, A f_\xi \rangle + \rho \sum_{\xi' \in I} a(\xi, \xi') \langle X_{\xi'} - X_\xi, f_\xi \rangle + \langle sS(X_\xi) + rR(X_\xi), f_\xi \rangle,$$

and let  $\langle b(X), \mathbf{f} \rangle := \sum_{\xi \in I} \langle b_\xi(X), f_\xi \rangle$ . The generator  $\mathcal{L}_{s,r,\rho}$  can then be written as

$$\mathcal{L}_{s,r,\rho} F(X) = \left\langle b(X), \frac{\delta F}{\delta X} \right\rangle + \frac{1}{2} \sum_{\xi \in I} \int_E \int_E \frac{\delta^2 F}{\delta X_\xi(u) \delta X_\xi(v)} Q_{X_\xi}(du, dv), \quad (2.4)$$

where  $\frac{\delta F}{\delta X} = \left( \frac{\delta F}{\delta X_\xi} \right)_{\xi \in I}$ .

**Theorem 2.1.** *For each  $X$  in  $M_1(E)^I$ , the martingale problem associated with generator  $(\mathcal{L}_{s,r,\rho}, A)$  starting at  $X$  is well-posed.*

**Proof:** The case of  $\rho = 0$ , and the case of  $A = 0, s = r = 0$  can be found respectively in [5] and [1]. The case of  $r = 0$  was obtained in [7]. The general case was studied in [2], where the index set  $I$  is either the finite dimensional lattice or the hierarchical group, and the type space is the set of integers.

Even though the index set and state space in our model are more general, the proofs are similar to that used in [7] and [2]. For completeness, we sketch a proof below.

Following [6], define the following system of Wright-Fisher type Markov chains. For each colony  $\xi$  in  $I$ , consider a population of  $N$  individuals with types in the space  $E$ . The population evolves under the influence of mutation, selection, recombination, migration, and genetic drift. Future generations are formed as follows: each individual chooses a pair in the current generation as parents. The probability that a particular pair is chosen is weighted by the fitness (described by  $V(x, y)$ ) of the pair. After the parents are selected, a recombination of the parent types occurs. The type created through recombination will change again: first through migration and then mutation. Existence for the martingale problem follows from the tightness of the empirical processes of approximating systems of Markov chains.

Uniqueness follows from the existence of a dual process. Let

$$\mathcal{H} := \bigcup_{m=1}^{\infty} (C(E^m) \times I^m).$$

For each solution  $X(t) = (X_{\xi}(t))$  to the martingale problem associated with  $\mathcal{L}_{s,r,\rho}$ , the law of  $X(t)$  is determined by

$$F((f, \pi), X(t)) = E^{X(0)} \left( \int_E \cdots \int_E f(u_1, \dots, u_m) X_{\xi_1}(du_1) \cdots X_{\xi_m}(du_m) \right)$$

for all  $(f, \pi)$  in  $C(E^m) \times I^m$ ,  $m \geq 1$ .

For  $F(X) = \prod_{i=1}^m \langle X_{\xi_i}, f_i \rangle$  in  $\mathcal{A}$ , direct calculations give

$$\begin{aligned} \mathcal{L}_{s,r,\rho} F(X) &= \sum_{i=1}^m \left\{ \langle X_{\xi_i}, Af_i \rangle + s \langle S(X_{\xi_i}) + rR(X_{\xi_i}), f_i \rangle + \rho \sum_{\xi' \in I} a(\xi_i, \xi') \langle X_{\xi'} - X_{\xi_i}, f_i \rangle \right\} \prod_{j \neq i} \langle X_{\xi_j}, f_j \rangle \\ &+ \sum_{1 \leq i < k \leq m, \xi_i = \xi_k} (\langle X_{\xi_i}, f_i f_k \rangle - \langle X_{\xi_i}, f_i \rangle \langle X_{\xi_k}, f_k \rangle) \prod_{j \neq i, k} \langle X_{\xi_j}, f_j \rangle. \end{aligned} \quad (2.5)$$

Define for  $\pi = (\xi_1, \dots, \xi_m)$  in  $I^m$ ,  $m \geq 1$  and  $f(u_1, \dots, u_m) = \prod_{i=1}^m f_i(u_i)$

$$\begin{aligned} X_{\pi}(du_1, \dots, du_m) &:= \prod_{i=1}^m X_{\xi_i}(du_i), \\ \tilde{\pi}^i &:= (\xi_1, \dots, \xi_m, \xi_i), i = 1, \dots, m, \\ \tilde{\pi}^{ii} &:= (\xi_1, \dots, \xi_m, \xi_i, \xi_i), i = 1, \dots, m, \\ \hat{\pi}^j &:= (\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_m), j = 2, \dots, m, \\ \pi^{i,\xi} &:= (\xi_1, \dots, \xi_{i-1}, \xi, \xi_{i+1}, \dots, \xi_m), \end{aligned}$$

and

$$\begin{aligned} A^m f(u_1, \dots, u_m) &:= \sum_{i=1}^m Af_i(u_i) \prod_{j \neq i} f_j(u_j), \\ H_{im} f(u_1, \dots, u_m) &:= (V(u_i, u_m) - V(u_m, u_{m+1})) f(u_1, \dots, u_m), \\ K_i f(u_1, \dots, u_m) &:= \int_E f(u_1, \dots, \nu, u_{i+1}, \dots, u_m) \eta(u_i, u_{m+1}; d\nu) - f(u_1, \dots, u_m). \end{aligned}$$

Then (2.5) can be written as

$$\begin{aligned} \mathcal{L}_{s,r,\rho} F(X) &= \langle X_{\pi}, A^m f \rangle \\ &+ \sum_{i=1}^m \left\{ s \langle X_{\tilde{\pi}^{ii}}, H_{im} f \rangle + r \langle X_{\tilde{\pi}^i}, K_i f \rangle + \rho \sum_{\xi' \in I} a(\xi_i, \xi') \langle X_{\tilde{\pi}^{i,\xi'}} - X_{\pi}, f \rangle \right\} \\ &+ \sum_{1 \leq i < k \leq m} (\langle X_{\tilde{\pi}^k}, \Phi_{ik} f \rangle - \langle X_{\pi}, f \rangle), \end{aligned}$$

where  $\Phi_{ik}f$  is the function in  $B(E^{m-1})$  that is obtained from  $f$  by replacing  $u_k$  with  $u_i$  and relabeling the variables.

The dual process  $(f_t, \pi_t)$  is an  $\mathcal{H}$ -valued process, starting from  $(f_0, \pi_0) = (f, \pi)$ , that involves the following transitions:

- coordinates of  $\pi_t$  are independent continuous time Markov chains on  $I$  with transition rate  $(\rho a(\xi, \xi'))_{\xi, \xi' \in I}$ ;
- any two coordinates of  $\pi_t$  that are the same will coalesce into one element at the same site with rate one;
- at rate  $s$  a coordinate of  $\pi_t$  will create two copies of itself so that the size of  $\pi_t$  is increased by two;
- at rate  $r$  a coordinate of  $\pi_t$  will create a copy of itself so that the size of  $\pi_t$  is increased by one;
- $f_0$  is in  $C(E^{|\pi_0|})$ ; between transitions of  $\pi_t$ ,  $f_t$  follows a deterministic path determined by the semigroup associated with  $|\pi_t|$  independent copies of  $A$ -motion;
- At the time of coalescence, the corresponding variables in  $f_t$  are set equal, which results in a jump from space  $C(E^{|\pi_t-|})$  to space  $C(E^{|\pi_t-|-1})$ ;

- If two new coordinates are created when the current number of variables is  $m$ , then we have

$$f(u_1, \dots, u_m) \rightarrow s(V(u_i, u_{m+1}) - V(u_{m+1}, u_{m+2}))f(u_1, \dots, u_m);$$

- If one new coordinate is created when the current number of variables is  $m$ , then we have

$$f(u_1, \dots, u_m) \rightarrow \int_E f(u_1, \dots, u_{i-1}, \nu, u_{i+1}, \dots, u_m) \eta(u_i, u_{m+1}; d\nu).$$

The uniqueness now follows from the following duality relation

$$E_{X(0)}[\langle X_\pi(t), f \rangle] = E_{(f, \pi)}[\langle X_{\pi_t}(0), f_t \rangle e^{s \int_0^t |\pi_u| du}].$$

□

### 3 Quasi-invariance and the cocycle identity

In this section we prove the main result of the paper relating the reversibility of probability measures on  $M_1(E)$  with their quasi-invariance. These results generalize those proved by Handa for the single site Fleming-Viot process. In the sections that follow, we will show that reversibility is a very restrictive condition that only applies to very special cases of the Fleming-Viot model.

**Definition 3.1.** A probability measure  $\Pi$  on  $M_1(E)$  is reversible for the Fleming-Viot operator  $(\mathcal{L}_{s,r,\rho}, \mathcal{A})$  if for  $\Phi, \Psi \in \mathcal{A}$ ,

$$\int \mathcal{L}_{s,r,\rho} \Phi(X) \Psi(X) \Pi(dX) = \int \mathcal{L}_{s,r,\rho} \Psi(X) \Phi(X) \Pi(dX).$$

For each  $\mathbf{f}$  in  $C(E)^I$ , define a map  $S_{\mathbf{f}} : M_1(E)^I \rightarrow M_1(E)^I$  by  $S_{\mathbf{f}}(X) = (X_{\xi}^{f_{\xi}})_{\xi \in I}$ , where

$$X_{\xi}^{f_{\xi}}(dv) := \frac{e^{f_{\xi}(v)} X_{\xi}(dv)}{\langle X_{\xi}, e^{f_{\xi}} \rangle}.$$

It follows from the definition that  $S_{\mathbf{f}}(S_{\mathbf{g}}) = S_{\mathbf{f}+\mathbf{g}}$  for any  $\mathbf{f}, \mathbf{g}$  in  $C(E)^I$ . For any  $\mathbf{f}$  in  $C(E)^I$  and probability measure  $\Pi$  on  $M_1(E)^I$ , set  $\Pi^{\mathbf{f}}(\cdot) := \Pi(S_{\mathbf{f}}(\cdot))$ .

The probability  $\Pi$  is called *quasi-invariant* for  $\mathcal{D}(A)_0^I$  if for any  $\mathbf{f} \in \mathcal{D}(A)_0^I$ , the measures  $\Pi^{\mathbf{f}}$  and  $\Pi$  are mutually absolutely continuous with

$$\frac{d\Pi^{\mathbf{f}}}{d\Pi}(X) = \exp\{\Lambda(\mathbf{f}, X)\},$$

where  $\Lambda : \mathcal{D}(A)_0^I \times M_1(E)^I \mapsto \mathbb{R}$  is called the *cocycle* associated with  $\Pi$ .

A direct result of the quasi-invariance is the following *cocycle identity*: for any  $\mathbf{f}, \mathbf{g} \in \mathcal{D}(A)_0^I$ , for  $\Pi$  almost all  $X$ ,

$$\Lambda(\mathbf{f} + \mathbf{g}, X) = \Lambda(\mathbf{f}, S_{\mathbf{g}}(X)) + \Lambda(\mathbf{g}, X). \quad (3.1)$$

The *carré du champ* associated with the operator  $\mathcal{L}_{s,r,\rho}$  is defined by

$$\Gamma(\Phi, \Psi) = \frac{1}{2}(\mathcal{L}_{s,r,\rho}(\Phi\Psi) - \Phi\mathcal{L}_{s,r,\rho}(\Psi) - \mathcal{L}_{s,r,\rho}(\Phi)\Psi), \quad \Phi, \Psi \in \mathcal{A}. \quad (3.2)$$

For any two functions  $f, g$  in  $B(E)$ , set  $(f \otimes g)(u, v) := f(u)g(v)$ . By an argument similar to that used in the proof of Lemma 3.1 in [8], we obtain the following result.

**Lemma 3.1.** For  $\Phi, \Psi \in \mathcal{A}$  and  $X \in M_1(E)^I$ ,

$$\Gamma(\Phi, \Psi)(X) = \frac{1}{2} \sum_{\xi \in I} \left\langle Q_{X_{\xi}}, \frac{\delta\Phi(X)}{\delta X_{\xi}} \otimes \frac{\delta\Psi(X)}{\delta X_{\xi}} \right\rangle, \quad (3.3)$$

and for  $\Phi, \Psi_1, \Psi_2 \in \mathcal{A}$ ,

$$\Gamma(\Phi\Psi_1, \Psi_2) + \Gamma(\Phi\Psi_2, \Psi_1) - \Gamma(\Phi, \Psi_1\Psi_2) = 2\Phi\Gamma(\Psi_1, \Psi_2). \quad (3.4)$$

**Lemma 3.2.** The probability measure  $\Pi$  is reversible with respect to  $\mathcal{L}_{s,r,\rho}$  if and only if

$$-\frac{1}{2} \int \left\langle Q_{X_{\xi}}, \frac{\delta\Phi(X)}{\delta X_{\xi}} \otimes f_{\xi} \right\rangle \Pi(dX) = \int \Phi(X) \langle b_{\xi}(X), f_{\xi} \rangle \Pi(dX) \quad (3.5)$$

for any  $\Phi \in \mathcal{A}$ ,  $\xi \in I$ , and  $f_{\xi} \in \mathcal{D}(A)$ .

**Proof:** Assume that  $\Pi$  is reversible with respect to  $\mathcal{L}_{s,r,\rho}$ . For a fixed  $\xi$  in  $I$ , let  $\Psi(X) = \langle X_\xi, f_\xi \rangle$ . It follows from (2.4) that  $\mathcal{L}_{s,r,\rho}\Psi(X) = \langle b_\xi(X), f_\xi \rangle$ . This, combined with Lemma 3.1 and reversibility, implies (3.5).

Next we assume that (3.5) holds. First we show, by induction on  $n$ , that for any  $n \geq 1$

$$\int \Phi(X) \mathcal{L}_{s,r,\rho} \Psi^{(n)}(X) \Pi(dX) = - \int \Gamma(\Phi, \Psi^{(n)})(X) \Pi(dX), \quad (3.6)$$

for any  $\Phi \in \mathcal{A}$ ,  $f_i \in \mathcal{D}(A)$ ,  $\xi_i \in I$ ,  $i = 1, \dots, n$ , and

$$\Psi^{(n)}(X) := \prod_{i=1}^n \Psi_i(X) := \prod_{i=1}^n \langle X_{\xi_i}, f_i \rangle.$$

The case of  $n = 1$  follows from (3.3) and (3.5). Assume that (3.6) holds for  $n \leq k$ . It follows from (3.2) and (3.4) that

$$\begin{aligned} \Phi \mathcal{L}_{s,r,\rho}(\Psi^{(k+1)}) &= \Phi [2\Gamma(\Psi^{(k)}, \Psi_{k+1}) + \Psi_{k+1} \mathcal{L}_{s,r,\rho}(\Psi^{(k)}) + \Psi^{(k)} \mathcal{L}_{s,r,\rho} \Psi_{k+1}] \\ &= \Gamma(\Phi \Psi^{(k)}, \Psi_{k+1}) + \Phi \Psi^{(k)} \mathcal{L}_{s,r,\rho} \Psi_{k+1} \\ &\quad + \Gamma(\Phi \Psi_{k+1}, \Psi^{(k)}) + \Phi \Psi_{k+1} \mathcal{L}_{s,r,\rho}(\Psi^{(k)}) \\ &\quad - \Gamma(\Phi, \Psi^{(k+1)}) \end{aligned} \quad (3.7)$$

which implies that

$$\int \Phi(X) \mathcal{L}_{s,r,\rho}(\Psi^{(k+1)})(X) \Pi(dX) = - \int \Gamma(\Phi, \Psi^{(k+1)})(X) \Pi(dX).$$

It follows from (3.6) that for any  $\Phi, \Psi$  in  $\mathcal{A}$

$$\int \Phi(X) \mathcal{L}_{s,r,\rho} \Psi(X) \Pi(dX) = - \int \Gamma(\Psi, \Phi)(X) \Pi(dX);$$

and by symmetry

$$\int \Psi(X) \mathcal{L}_{s,r,\rho} \Phi(X) \Pi(dX) = \int \Phi(X) \mathcal{L}_{s,r,\rho} \Psi(X) \Pi(dX).$$

Therefore,  $\Pi$  is reversible with respect to  $\mathcal{L}_{s,r,\rho}$ .  $\square$

**Lemma 3.3.** *Suppose  $\mathbf{f} \in C(E)^I$  and put  $X_t := S_{-t} X$  for  $X \in M_1(E)^I$  and  $t \in \mathbb{R}$ . For every  $\Phi \in \tilde{\mathcal{A}}$  we have*

$$\frac{d}{dt} \Phi(X_t) = - \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \Phi(X_t)}{\delta X_\xi} \right\rangle. \quad (3.8)$$

**Proof:** Since both sides of the equation are linear, it suffices to prove the result for functions of the form  $\Phi(X) = \prod_{i=1}^m \langle X_{\xi_i}, g_i \rangle$ , where  $m$  a positive integer,  $(\xi_i)_{1 \leq i \leq m}$  in  $I$ , and  $g_i \in B(E)$ . But both sides of the equation are also derivations in  $\Phi$ , so it suffices to take  $m = 1$ . But in this case, (3.8) follows from an easy calculation or Lemma 3.3 of [8].  $\square$

For  $\mathbf{f} \in \mathcal{D}(A)_0^I$  and  $X \in M_1(E)^I$ , we let

$$\Lambda(\mathbf{f}, X) := 2 \int_0^1 \langle b(S_{s\mathbf{f}}X), \mathbf{f} \rangle ds. \quad (3.9)$$

**Lemma 3.4.** *Suppose  $\mathbf{f} \in \mathcal{D}(A)_0^I$ , and put  $X_t = S_{-t\mathbf{f}}X$  for  $X \in M_1(E)^I$  and  $t \in \mathbb{R}$ . Then we can write*

$$\Lambda(t\mathbf{f}, X_t) = 2 \int_0^t \langle b(X_s), \mathbf{f} \rangle ds.$$

**Proof:**

$$\begin{aligned} \Lambda(t\mathbf{f}, X_t) &= 2 \int_0^1 \langle b(S_{st\mathbf{f}}X_t), t\mathbf{f} \rangle ds = 2t \int_0^1 \langle b(S_{-(1-s)t\mathbf{f}}X), \mathbf{f} \rangle ds \\ &= 2t \int_0^1 \langle b(S_{-st\mathbf{f}}X), \mathbf{f} \rangle ds = 2 \int_0^t \langle b(S_{-s\mathbf{f}}X), \mathbf{f} \rangle ds. \end{aligned}$$

□

The following lemma proves formula (3.8) for certain functions  $F \notin \tilde{\mathcal{A}}$ .

**Lemma 3.5.** *Suppose  $\mathbf{f} \in C(E)_0^I$ , and put  $X_t = S_{-t\mathbf{f}}X$  for  $X \in M_1(E)^I$  and  $t \in \mathbb{R}$ . For  $h \in C(E)$  and the sequence  $c(\xi)$  satisfying  $\sum_{\xi \in I} |c(\xi)| < \infty$ , define  $F : M_1(E)^I \rightarrow \mathbb{R}$  by*

$$F(X) := \left\langle \sum_{\xi \in I} c(\xi) X_\xi, h \right\rangle.$$

Then

$$\frac{d}{dt} F(X_t) = - \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta F(X_t)}{\delta X_\xi} \right\rangle.$$

**Proof:** Let  $I_0$  be a finite subset of  $I$  such that  $f_\xi = 0$  for  $\xi \notin I_0$ . Define

$$F_0(X) := \left\langle \sum_{\xi \in I_0} c(\xi) X_\xi, h \right\rangle.$$

Clearly  $F_0 \in \tilde{\mathcal{A}}$ . Also,  $(X_t)_\xi = X_\xi$  for  $\xi \notin I_0$  so those terms have a zero time derivative. Therefore,

$$\frac{d}{dt} F(X_t) = \frac{d}{dt} F_0(X_t).$$

It follows from direct calculation that

$$\frac{\delta F_0(X_t)}{\delta X_\xi} = \begin{cases} \frac{\delta F(X_t)}{\delta X_\xi} & \text{if } \xi \in I_0, \\ 0 & \text{if } \xi \notin I_0. \end{cases}$$

Since  $f_\xi \equiv 0$  for  $\xi \notin I_0$ , this gives

$$\begin{aligned} \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta F(X_t)}{\delta X_\xi} \right\rangle &= \sum_{\xi \in I_0} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta F(X_t)}{\delta X_\xi} \right\rangle \\ &= \sum_{\xi \in I_0} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta F_0(X_t)}{\delta X_\xi} \right\rangle \\ &= \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta F_0(X_t)}{\delta X_\xi} \right\rangle, \end{aligned}$$

which, combined with Lemma 3.3, implies the result.  $\square$

**Theorem 3.1.** *If the probability measure  $\Pi$  in  $M_1(M_1(E)^I)$  is reversible for  $\mathcal{L}_{s,r,\rho}$ , then  $\Pi$  is quasi-invariant for  $\mathcal{D}(A)_0^I$  with cocycle  $\Lambda(\mathbf{f}, X)$  given by (3.9).*

**Proof:** Assume that  $\Pi \in M_1(M_1(E)^I)$  is reversible with respect to  $\mathcal{L}_{s,r,\rho}$ , and fix  $\mathbf{f} \in \mathcal{D}(A)_0^I$ . We must show that

$$\int F(X) (S_{\mathbf{f}}\Pi)(dX) = \int F(S_{-\mathbf{f}}X) \Pi(dX) = \int F(X) e^{\Lambda(\mathbf{f}, X)} \Pi(dX),$$

for sufficiently many functions  $F : M_1(E)^I \rightarrow \mathbb{R}$ . Since  $\exp(-\Lambda(\mathbf{f}, X))$  is strictly positive and  $\mathcal{A}$  is measure determining, it suffices to prove that for any  $\Phi \in \mathcal{A}$

$$\int \Phi(S_{-\mathbf{f}}X) e^{-\Lambda(\mathbf{f}, S_{-\mathbf{f}}X)} \Pi(dX) = \int \Phi(X) \Pi(dX).$$

In what follows we shall show that

$$Z(t) := \int \Phi(S_{-t\mathbf{f}}X) e^{-\Lambda(t\mathbf{f}, S_{-t\mathbf{f}}X)} \Pi(dX)$$

is a constant function of  $t \in \mathbb{R}$ . Setting

$$\tilde{\Phi}_t(X) := \Phi(X_t) e^{-\Lambda(t\mathbf{f}, X_t)} = \Phi(S_{-t\mathbf{f}}X) e^{-\Lambda(t\mathbf{f}, S_{-t\mathbf{f}}X)},$$

and noting that  $\Lambda(t\mathbf{f}, X_t) = 2 \int_0^t \langle b(X_s), \mathbf{f} \rangle ds$ , we have

$$\frac{\delta \tilde{\Phi}_t(X)}{\delta X_\xi}(u) = \frac{\delta \Phi(X_t)}{\delta X_\xi}(u) e^{-\Lambda(t\mathbf{f}, X_t)} - 2\tilde{\Phi}_t(X) \int_0^t \frac{\delta \langle b(X_s), \mathbf{f} \rangle}{\delta X_\xi}(u) ds. \quad (3.10)$$

It follows that

$$\begin{aligned} &\sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \tilde{\Phi}_t(X)}{\delta X_\xi} \right\rangle \\ &= \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \Phi(X_t)}{\delta X_\xi} \right\rangle e^{-\Lambda(t\mathbf{f}, X_t)} - 2\tilde{\Phi}_t(X) \int_0^t \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \langle b(X_s), \mathbf{f} \rangle}{\delta X_\xi} \right\rangle ds \\ &= -\frac{d}{dt} \Phi(X_t) e^{-\Lambda(t\mathbf{f}, X_t)} + 2\tilde{\Phi}_t(X) \int_0^t \frac{d}{ds} \langle b(X_s), \mathbf{f} \rangle ds \\ &= -\frac{d}{dt} \Phi(X_t) e^{-\Lambda(t\mathbf{f}, X_t)} + 2\tilde{\Phi}_t(X) (\langle b(X_t), \mathbf{f} \rangle - \langle b(X), \mathbf{f} \rangle), \end{aligned}$$

where Lemmas 3.3 and 3.5 are used for obtaining the second equality. Therefore,

$$\begin{aligned}
Z'(t) &= \int \left( \frac{d}{dt} \Phi(X_t) e^{-\Lambda(t\mathbf{f}, X_t)} + \Phi(X_t) \frac{d}{dt} e^{-\Lambda(t\mathbf{f}, X_t)} \right) \Pi(dX) \\
&= \int \left( - \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \tilde{\Phi}_t(X)}{\delta X_\xi} \right\rangle + 2\tilde{\Phi}_t(X) (\langle b(X_t), \mathbf{f} \rangle - \langle b(X), \mathbf{f} \rangle) - 2\tilde{\Phi}_t(X) \langle b(X_t), \mathbf{f} \rangle \right) \Pi(dX) \\
&= - \int \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \tilde{\Phi}_t(X)}{\delta X_\xi} \right\rangle \Pi(dX) - 2 \int \tilde{\Phi}_t(X) \langle b(X), \mathbf{f} \rangle \Pi(dX).
\end{aligned}$$

By reversibility and Lemma 3.2,

$$\int \sum_{\xi \in I} \left\langle Q_{X_\xi}, f_\xi \otimes \frac{\delta \Phi(X)}{\delta X_\xi} \right\rangle \Pi(dX) + 2 \int \langle b(X), \mathbf{f} \rangle \Phi(X) \Pi(dX) = 0, \quad (3.11)$$

for  $\Phi \in \mathcal{A}$ . In the Appendix, we introduce a space of functions  $\mathcal{H}$  that contains  $\mathcal{A}$ , and show that  $\tilde{\Phi}_t(X) \in \mathcal{H}$  and (3.11) holds for all  $\Phi$  in  $\mathcal{H}$ . These imply that  $Z'(t) = 0$ . Therefore,  $Z(1) = Z(0)$  and the theorem follows from

$$\int \Phi(S_{-\mathbf{f}}X) e^{-\Lambda(\mathbf{f}, S_{-\mathbf{f}}X)} \Pi(dX) = Z(1) = Z(0) = \int \Phi(X) \Pi(dX).$$

□

**Theorem 3.2.** *If the probability measure  $\Pi$  in  $M_1(M_1(E)^I)$  is quasi-invariant with cocycle given by (3.9), then  $\Pi$  is reversible with respect to  $\mathcal{L}_{s,r,\rho}$ .*

**Proof:** Suppose that  $\Pi$  is quasi-invariant with cocycle given by (3.9). Then for any  $\xi \in I$  and  $\mathbf{f}$  in  $C(E)^I$  such that  $f_\xi \in \mathcal{D}(A)$  and  $f_{\xi'} = 0$  for  $\xi' \neq \xi$ , the function

$$Z(t) = \int \Phi(S_{-t\mathbf{f}}X) e^{-\Lambda(t\mathbf{f}, S_{-t\mathbf{f}}X)} \Pi(dX)$$

is constant in  $t \in \mathbb{R}$ . Noting that

$$0 = Z'(0) = - \int \sum_{\xi \in I} \left\langle f_\xi \otimes \frac{\delta \Phi(X)}{\delta X_\xi}, Q_{X_\xi} \right\rangle \Pi(dX) - 2 \int \Phi(X) \langle b(X), \mathbf{f} \rangle \Pi(dX),$$

and  $f_\xi$  is arbitrary in  $\mathcal{D}(A)$ , the theorem follows from Lemma 3.2. □

## 4 Consequences of the cocycle identity

It follows from the cocycle identity (3.1) that for any  $X$  in  $M_1(E)^I$  and any  $\mathbf{f}, \mathbf{g} \in \mathcal{D}(A)_0^I$ ,

$$\Lambda(\mathbf{f}, S_{\mathbf{g}}(X)) - \Lambda(\mathbf{f}, X) = \Lambda(\mathbf{g}, S_{\mathbf{f}}(X)) - \Lambda(\mathbf{g}, X). \quad (4.1)$$

For any two distinct  $\xi_1, \xi_2$  in  $I$ , and  $f, g$  in  $\mathcal{D}(A)$ , let  $\mathbf{f} = (f_\xi)$  and  $\mathbf{g} = (g_\xi)$  be such that  $f_{\xi_1} = f, f_\xi = 0$  for  $\xi \neq \xi_1$ , and  $g_{\xi_2} = g, g_\xi = 0$  for  $\xi \neq \xi_2$ . By direct calculation,

$$\begin{aligned}\Lambda(\mathbf{f}, X) &= 2 \int_0^1 \{ \langle S_{u\mathbf{f}}(X)_{\xi_1}, Af \rangle + s \langle S(S_{u\mathbf{f}}(X)_{\xi_1}), f \rangle \\ &\quad + r \langle R(S_{u\mathbf{f}}(X)_{\xi_1}), f \rangle + \rho \sum_{\xi'} a(\xi_1, \xi') \langle S_{u\mathbf{f}}(X)_{\xi'} - S_{u\mathbf{f}}(X)_{\xi_1}, f \rangle \} du \\ &= 2 \int_0^1 \{ \langle X_{\xi_1}^{uf}, Af \rangle + s \langle S(X_{\xi_1}^{uf}), f \rangle \\ &\quad + r \langle R(X_{\xi_1}^{uf}), f \rangle + \rho \sum_{\xi' \neq \xi_1} a(\xi_1, \xi') \langle X_{\xi'} - X_{\xi_1}^{uf}, f \rangle \} du,\end{aligned}$$

and

$$\begin{aligned}\Lambda(\mathbf{f}, S_{\mathbf{g}}(X)) &= 2 \int_0^1 \{ \langle S_{u\mathbf{f}+\mathbf{g}}(X)_{\xi_1}, Af \rangle + s \langle S(S_{u\mathbf{f}+\mathbf{g}}(X)_{\xi_1}), f \rangle \\ &\quad + r \langle R(S_{u\mathbf{f}+\mathbf{g}}(X)_{\xi_1}), f \rangle + \rho \sum_{\xi'} a(\xi_1, \xi') \langle S_{u\mathbf{f}+\mathbf{g}}(X)_{\xi'} - S_{u\mathbf{f}+\mathbf{g}}(X)_{\xi_1}, f \rangle \} du \\ &= 2 \int_0^1 \{ \langle X_{\xi_1}^{uf}, Af \rangle + s \langle S(X_{\xi_1}^{uf}), f \rangle + r \langle R(X_{\xi_1}^{uf}), f \rangle \\ &\quad + \rho \sum_{\xi' \neq \xi_1, \xi_2} a(\xi_1, \xi') \langle X_{\xi'} - X_{\xi_1}^{uf}, f \rangle + \rho a(\xi_1, \xi_2) \langle X_{\xi_2}^g - X_{\xi_1}^{uf}, f \rangle \} du,\end{aligned}$$

which leads to

$$\Lambda(\mathbf{f}, S_{\mathbf{g}}(X)) - \Lambda(\mathbf{f}, X) = 2\rho a(\xi_1, \xi_2) \langle X_{\xi_2}^g - X_{\xi_1}^{uf}, f \rangle. \quad (4.2)$$

Together, (4.1) and (4.2) implies that for  $\rho > 0$

$$a(\xi_1, \xi_2) \langle X_{\xi_2}^g - X_{\xi_1}^{uf}, f \rangle = a(\xi_2, \xi_1) \langle X_{\xi_1}^f - X_{\xi_1}, g \rangle. \quad (4.3)$$

Let

$$\hat{I} := \{ \xi \in I : \text{there exists } \eta \in I, \text{ such that } a(\eta, \xi) > 0 \}.$$

It follows from (2.1) that the set  $\hat{I}$  is not empty.

**Lemma 4.1.** *Suppose that  $\Pi$  is a reversible probability measure with respect to  $\mathcal{L}_{s,r,\rho}$  with  $\rho > 0$ . Then for any  $\xi \in \hat{I}$ ,  $X_\xi$  is a Dirac measure with  $\Pi$  probability one, i.e.,*

$$\Pi\{X_\xi \in \Delta\} = 1. \quad (4.4)$$

**Proof:** Let  $C$  be a countable dense subset of  $E$ . By definition, for each  $\xi$  in  $\hat{I}$ , there exists  $\xi'$  in  $I$  such that  $a(\xi', \xi) > 0$ . Assume that with positive  $\Pi$  probability,  $X_\xi$  is not a Dirac measure. For any two

distinct elements  $c_1, c_2$  in  $C$ , and any positive rational numbers  $r_1, r_2$  satisfying  $r_1 + r_2 < d(c_1, c_2)$ , let

$$D(c_1, c_2; r_1, r_2) := \{X \in M_1(E)^I : X_\xi(B(c_1, r_1)) > 0, X_\xi(B(c_2, r_2)) > 0\},$$

where  $B(c_i, r_i)$  denotes the open ball in  $E$  with center  $c_i$  and radius  $r_i$ . Clearly,

$$\bigcup_{c_1, c_2; r_1, r_2} D(c_1, c_2, r_1, r_2) = \{X \in M_1(E)^I : X_\xi \neq \delta_u, \forall u \in E\}.$$

Therefore, we can find rational numbers  $c_1, c_2, r_1, r_2$  such that  $\Pi\{D(c_1, c_2, r_1, r_2)\} > 0$ . Choose a nonnegative continuous function  $f$  such that  $f(x) = 0$  for  $x \in B(c_1, r_1)$  and  $f(x) > 0$  for  $x \in B(c_2, r_2)$ . For any  $X_\xi \in D(c_1, c_2; r_1, r_2)$ , observe that  $\langle X_\xi, e^f \rangle > 1$ . When the signed measure  $X_\xi - X_\xi^f$  is restricted to set  $B(c_1, r_1)$ , we have

$$X_\xi - X_\xi^f = (1 - \langle X_\xi, e^f \rangle^{-1})X_\xi,$$

which is a measure on  $B(c_1, r_1)$  with strictly positive total mass. Let  $g$  be any continuous function such that  $g(x) > 0$  for  $x \in B(c_1, r_1)$  and  $g(x) = 0$  for  $x \notin B(c_1, r_1)$ .

For any  $h \in C(E)$  and any positive integer  $k$ , define

$$h^{(k)} := k \int_0^{\frac{1}{k}} P_s h ds.$$

Then  $\|h^{(k)} - h\|_\infty \rightarrow 0$ ,  $h^{(k)} \in \mathcal{D}(A)$  and  $Ah^{(k)} = k(P_{1/k}h - h) \in C(E)$ .

By dominated convergence theorem, we have

$$\lim_{k \rightarrow \infty} \langle X_\xi - X_\xi^{f^{(k)}}, ng^{(k)} \rangle = \langle X_\xi - X_\xi^f, ng \rangle = \langle X_\xi - X_\xi^f, ng 1_{B(c_1, r_1)} \rangle$$

and

$$\lim_{k \rightarrow \infty} \langle X_{\xi'} - X_{\xi'}^{ng^{(k)}}, f^{(k)} \rangle = \langle X_{\xi'} - X_{\xi'}^{ng}, f \rangle \leq \|f\|_\infty$$

for all  $n$ . Choosing  $\xi' = \xi_2, \xi = \xi_1, g = ng^{(k)}$  in (4.3), and taking the limit in the order of  $k \rightarrow \infty$  and  $n \rightarrow \infty$ , gives a contradiction.  $\square$

**Remark.** It follows from the above theorem that for each  $\xi$  in  $\hat{I}$ , there is a random variable  $x_\xi$  taking values in  $E$  such that  $X_\xi = \delta_{x_\xi}$  almost surely under  $\Pi$ .

**Lemma 4.2.** *Suppose that  $\Pi$  is a reversible measure with respect to  $\mathcal{L}_{s,r,\rho}$ . For each  $\xi$  in  $I$ , let  $I_\xi = \{\xi' \in I : a(\xi, \xi') > 0\}$ . Then for  $\xi \in \hat{I}$ , we have  $\Pi\{x_\xi = x_{\xi'}\} = 1$ , for all  $\xi' \in I_\xi$ .*

**Proof:** By Lemma 4.1, for  $\Pi$  almost all  $X$ , we have  $X_{\xi'} = \delta_{x_{\xi'}}$  for any  $\xi' \in I_\xi$ . For  $f, g \in \mathcal{D}(A)$ , set  $\Phi(X) = \langle X_\xi, f \rangle$  and  $\Psi(X) = \langle X_\xi, g \rangle$ . The reversibility, combined with Lemma 3.1, implies

$$-\int \Psi(X) \mathcal{L}_{s,r,\rho} \Phi(X) \Pi(dX) = \frac{1}{2} \int \langle f \otimes g, Q_{X_\xi} \rangle \Pi(dX) = 0, \quad (4.5)$$

since  $Q_{X_\xi}$  is the zero measure when  $X_\xi$  is a delta mass.

Then for any  $f, g \in \mathcal{D}(\mathcal{A})$  equation (4.5) gives

$$\int g(x_\xi) \left[ \sum_{\xi' \neq \xi} \rho a(\xi, \xi') (f(x_{\xi'}) - f(x_\xi)) + Af(x_\xi) + \tilde{R}f(x_\xi) \right] \Pi(dX) = 0, \quad (4.6)$$

where

$$\tilde{R}f(x) := r \left[ \int f(u) \eta(x, x; du) - f(x) \right], \quad x \in E.$$

For any  $c \in E$  and  $0 < r < r'$ , choose a sequence of continuous functions  $(f_m)$  on  $E$  such that  $0 \leq f_m \leq 1$  and  $f_m(x) = 1$  for  $x \in \bar{B}(c, r')$  and  $f_m$  converges, pointwisely, to  $1_{\bar{B}(c, r')}$ , where  $\bar{B}(c, r')$  denotes the closed ball with center  $c$  and radius  $r'$ ; also choose a sequence of continuous functions  $(g_n)$  on  $E$  such that  $0 \leq g_n \leq 1$ ,  $g_n(x) = 1$  for  $x \in \bar{B}(c, r)$ ,  $g_n$  has its support in  $\bar{B}(c, r')$ , and  $g_n$  converges pointwisely to  $1_{\bar{B}(c, r)}$ .

By the maximal principle for  $A$ , we have  $Af_m^{(k)}(x) \leq 0$  for  $x \in \bar{B}(c, r')$ , so that for  $m, n, k, k'$ ,

$$\int g_n^{(k')}(x_\xi) Af_m^{(k)}(x_\xi) \Pi(dX) \leq 0. \quad (4.7)$$

Since  $g_n^{(k')}$  converges pointwisely to  $g_n$  as  $k' \rightarrow \infty$  and  $f_m^{(k)}$  converges pointwisely to  $f_m$  as  $k \rightarrow \infty$ , taking limits in the order of  $k' \rightarrow \infty$ ,  $k \rightarrow \infty$ ,  $m \rightarrow \infty$ , and  $n \rightarrow \infty$ , we first have

$$\int g_n^{(k')}(x_\xi) \tilde{R}f_m^{(k)}(x_\xi) \Pi(dX) \rightarrow r \int 1_{\bar{B}(c, r)}(x_\xi) (\eta(x_\xi, x_\xi; \bar{B}(c, r')) - 1_{\bar{B}(c, r')}(x_\xi)) \Pi(dX) \leq 0, \quad (4.8)$$

then combining (4.6), (4.7) and (4.8) we further have

$$\int 1_{\bar{B}(c, r)}(x_\xi) \sum_{\xi' \neq \xi} a(\xi, \xi') (1_{\bar{B}(c, r')}(x_{\xi'}) - 1_{\bar{B}(c, r')}(x_\xi)) \Pi(dX) \geq 0. \quad (4.9)$$

Letting  $r' \rightarrow r+$  we have

$$\begin{aligned} & \int \sum_{\xi' \neq \xi} a(\xi, \xi') [1_{\bar{B}(c, r)}(x_\xi) 1_{\bar{B}(c, r)}(x_{\xi'}) - 1_{\bar{B}(c, r)}(x_\xi)] \Pi(dX) \\ &= \int 1_{\bar{B}(c, r)}(x_\xi) \sum_{\xi' \neq \xi} a(\xi, \xi') (1_{\bar{B}(c, r)}(x_{\xi'}) - 1_{\bar{B}(c, r)}(x_\xi)) \Pi(dX) \geq 0. \end{aligned} \quad (4.10)$$

Since

$$1_{\bar{B}(c, r)}(x_\xi) 1_{\bar{B}(c, r)}(x_{\xi'}) - 1_{\bar{B}(c, r)}(x_\xi) \leq 0,$$

it follows from (4.10) that for any  $\xi' \in I_\xi$

$$1_{\bar{B}(c, r)}(x_\xi) 1_{\bar{B}(c, r)}(x_{\xi'}) = 1_{\bar{B}(c, r)}(x_\xi),$$

$\Pi$  almost everywhere. Because  $c$  and  $r$  are arbitrary and  $E$  is separable, we have  $x_\xi = x_{\xi'}$   $\Pi$  almost everywhere.

## 5 Reversibility

Let  $\mathcal{L}$  denote the generator of the Fleming-Viot process with mutation, selection, recombination, and no migration on each colony.

**Definition 5.1.** *A generator  $A$  is said to be irreducible if for all  $x$  in  $E$  and any non-negative, non-zero measurable function  $g \in C(E)$ , there exists  $t > 0$  such that  $(P_t g)(x) > 0$ , where  $P_t$  is the semigroup generated by  $A$ .*

**Theorem 5.1.** *Assume that there is no migration, and the mutation generator  $A$  is irreducible. Let  $\Pi$  be the reversible measure for  $\mathcal{L}_{s,r,0}$ . Then for each  $\xi$  in  $I$ ,*

$$\Pi\{X \in M_1(E)^I : \text{supp}(X_\xi) = E\} = 1. \quad (5.1)$$

*A probability measure  $\Pi$  in  $M_1(M_1(E)^I)$  is reversible with respect to  $\mathcal{L}_{s,r,0}$  if and only if there are  $\theta > 0$ ,  $\mu$  in  $M_1(E)$ , and  $h$  in  $C(E)$  such that, for any  $g$  in  $C(E)$ , the mutation generator  $A$  and recombination kernel  $\eta(x, y; dz)$  satisfy*

$$\begin{aligned} Ag(x) + r \left[ \int g(z) \eta(x, x; dz) - g(x) \right] &= \frac{\theta}{2} [\langle \mu, g \rangle - g(x)], \\ \eta(x, y; dz) &= \frac{1}{2} (\eta(x, x; dz) + \eta(y, y; dz)) + (h(x) - h(y))(\delta_x(dz) - \delta_y(dz)). \end{aligned}$$

**Proof:** When there is no migration, the interacting system becomes a system of independent Fleming-Viot processes. The theorem is then a direct result of Proposition 3.1 and Theorem 2.2 in [8].  $\square$

**Theorem 5.2.** *Assume that  $\rho > 0$  and that  $E$  is not a one point space. If the mutation operator  $A$  is irreducible, then there is no reversible measure with respect to  $\mathcal{L}_{s,r,\rho}$ .*

**Proof:** Let  $\Pi$  be reversible for  $\mathcal{L}_{s,r,\rho}$ . For any  $\xi$  in  $\hat{I}$ , (4.4) shows that  $X_\xi$  is  $\Pi$ -almost surely a Dirac measure. On the other hand, the projection of  $\Pi$  to each colony  $\xi$  in  $\hat{I}$  is a reversible measure of the Fleming-Viot process on the colony. Applying Proposition 3.1 in [8] again it follows that  $X_\xi$  has full support  $\Pi$ -almost surely. This implies that  $E$  is a one point space. A contradiction.  $\square$

We now consider the case of zero mutation. For any  $\xi, \xi' \in I$ , write  $\xi' \rightarrow \xi$  if either  $a(\xi', \xi) > 0$  or there exists a finite sequence  $\xi_i, i = 1, \dots, n$  such that  $a(\xi', \xi_1) > 0, a(\xi_1, \xi_2) > 0, \dots, a(\xi_n, \xi) > 0$ . Recall that  $\Delta$  denotes the collection of Dirac measures on  $E$ . Set

$$\Delta_a := \{X \in M_1(E)^I : X_\xi = X_{\xi'} \in \Delta, \forall \xi, \xi' \in I \text{ with } \xi' \rightarrow \xi\}.$$

**Theorem 5.3.** *Suppose that  $\rho > 0$  and for any  $\xi$  in  $I$ , there is  $\xi'$  such that  $\xi' \rightarrow \xi$ . If there is no mutation or recombination, then  $\Pi$  is reversible if and only if its support is in  $\Delta_a$ .*

**Proof:** The necessity follows from Lemma 4.1 and Lemma 4.2. If the mutation and recombination are zero, then for any  $X \in \Delta_a$ , and any  $F \in \mathcal{A}$  we have  $\mathcal{L}_{s,r,\rho}F(X) = 0$ , which gives the sufficiency.  $\square$

## 6 Examples

In this section, we discuss the reversibility of several well-known examples.

**Example 1.** (Two Type Stepping-Stone Model). Let  $I = \mathbb{Z}^d$  be the  $d$  dimensional lattice, and  $E = \{0, 1\}$ . Let  $x_i$  denote the the proportion of type 0 individuals on colony  $i$  in  $\mathbb{Z}^d$ . The generator on colony  $i$  is given by

$$L = \frac{1}{2}a_i(x)\frac{\partial^2}{\partial x_i^2} + b_i(x)\frac{\partial}{\partial x_i},$$

where

$$\begin{aligned} x &= (x_i : i \in \mathbb{Z}^d), \quad a_i(x) = x_i(1 - x_i), \\ b_i(x) &= \sum_{j \in \mathbb{Z}^d} \alpha(i, j)(x_j - x_i) + v - (u + v)x_i + sx_i(1 - x_i), \\ \alpha(i, j) &\geq 0, u, v \geq 0. \end{aligned}$$

This is the model studied in [11] and [13]. It follows from Theorem 5.2 and Theorem 5.3 that the model has a reversible measure if and only if  $d \leq 2$  and  $u = v = 0$ .

**Example 2.** This model, studied in [1], has zero mutation and recombination. Let  $I$  be either  $\mathbb{Z}^d$  or the hierarchical group  $\Omega_N$ . In addition to assumption (2.1), the migration rate satisfies  $a(\xi, \xi') = a(0, \xi' - \xi)$  and  $\sum_{n=0}^{\infty} (a^n(0, \xi) + a^n(\xi, 0)) > 0$ . Set  $\hat{a}(\xi, \xi') = \frac{1}{2}[a(\xi, \xi') + a(\xi', \xi)]$ . Theorem 5.2 and Theorem 5.3 imply that the model has a reversible measure if and only if the symmetrized kernel  $\hat{a}$  is recurrent.

## 7 Appendix

**Definition.** Let  $S$  be a metric space. A sequence  $\{h_n\} \subset B(S)$  is said to converge boundedly and pointwise to  $f \in B(S)$  if  $h_n(x) \rightarrow f(x)$  for all  $x \in S$  and  $\sup_n \|h_n\|_{\infty} < \infty$ . We write

$$\text{bp} - \lim_{n \rightarrow \infty} h_n = f.$$

Part 1. The space  $\mathcal{H}$ .

Define  $\mathcal{H}$  to be the space of functions  $F : M_1(E)^I \rightarrow \mathbb{R}$  so that the partial derivative  $\delta F(X)/\delta X_\xi(u)$  exists for every  $X, \xi$ , and  $u$ , and (3.11) holds with  $\Phi$  replaced by  $F$ .

Our first observation is that for any positive integer  $m$ , any  $\mathbf{f} \in B(E^m)$  and any  $(\xi_1, \dots, \xi_m) \subset I^m$ , the function  $F_{\mathbf{f}} : M_1(E)^I \mapsto \mathbb{R}$  defined by  $F_{\mathbf{f}}(X) := \langle \otimes_{i=1}^m X_{\xi_i}, \mathbf{f} \rangle$  belongs to  $\mathcal{H}$ . First consider the case of  $\mathbf{f} = 1_{G_1 \times \dots \times G_m}$  for open sets  $G_i \subset E, i = 1, \dots, m$ . Since we can approximate the indicator function  $1_{G_i}$  boundedly and pointwise by functions in  $\mathcal{D}(A)$  which is dense in  $C(E)$ , it follows that one can find a sequence of functions  $\mathbf{f}_n$  in  $\mathcal{A}$  such that  $\text{bp} - \lim_{n \rightarrow \infty} \mathbf{f}_n = \mathbf{f}$ . Since the bp-convergence of  $\mathbf{f}_n$  to  $\mathbf{f}$  implies the bp-convergence of the corresponding derivatives, we have that  $\langle \otimes_{i=1}^m X_{\xi_i}, \mathbf{f} \rangle \in \mathcal{H}$ . Then the observation follows from Theorem 4.3 in the Appendixes of [4].

Using the above-mentioned observation and polynomial approximation we can further show that for any  $m_i$ , any  $(\xi_{i1}, \dots, \xi_{im_i}) \in I^{m_i}$ , any  $\mathbf{f}_i \in B(\mathbb{R}^{m_i}), i = 1, \dots, n$ , and any  $\phi \in C^1(\mathbb{R}^n)$ , the function  $F : M_1(E)^I \mapsto \mathbb{R}$  defined by

$$F(X) := \phi(\langle \otimes_{j=1}^{m_1} X_{\xi_{1j}}, \mathbf{f}_1 \rangle, \dots, \langle \otimes_{j=1}^{m_n} X_{\xi_{nj}}, \mathbf{f}_n \rangle)$$

also belongs to  $\mathcal{H}$ .

Moreover, take  $g = \otimes_{i=1}^m g_i$  with  $g_i \in \mathcal{D}(A)$  bounded below by  $c > 0$ , and  $k \in B(E^m)$  and set  $F(X) := \Phi^k(X)/\Phi^g(X)$ . By polynomial approximation again we can show that  $F \in \mathcal{H}$ .

Part 2. Approximating  $\tilde{\Phi}_t$ .

Let  $\mathbf{f} \in \mathcal{D}(A)_0^I$  such that outside the finite subset  $I_0$  of  $I$   $f_\xi \equiv 0$ , and  $X_s = S_{-s\mathbf{f}}X$ . Then

$$\begin{aligned} \langle b(X_s), \mathbf{f} \rangle &= \sum_{\xi \in I_0} \langle X_\xi^{sf_\xi}, Af_\xi \rangle + \rho \sum_{\xi \in I_0} \sum_{\xi' \in I} a(\xi, \xi') \langle X_{\xi'}^{sf_{\xi'}} - X_\xi^{sf_\xi}, f_\xi \rangle \\ &\quad + s \sum_{\xi \in I_0} \left( \int_E \int_E V(u, v) f_\xi(u) X_\xi^{sf_\xi}(dv) X_\xi^{sf_\xi}(du) - \langle f_\xi, X_\xi^{sf_\xi} \rangle \langle V, X_\xi^{sf_\xi \otimes 2} \rangle \right) \\ &\quad + r \sum_{\xi \in I_0} \left( \left\langle \int_E f_\xi(u) \eta(\cdot, \cdot; du), X_\xi^{sf_\xi \otimes 2} \right\rangle - \langle f_\xi, X_\xi^{sf_\xi} \rangle \right). \end{aligned}$$

Since  $\sum_{\xi' \in I} a(\xi, \xi') < \infty$  and

$$\sum_{\xi \in I_0} \sum_{\xi' \in I} a(\xi, \xi') \langle X_{\xi'}^{sf_{\xi'}}, f_\xi \rangle = \sum_{\xi \in I_0} \sum_{\xi' \in I_0} a(\xi, \xi') \langle X_{\xi'}^{sf_{\xi'}}, f_\xi \rangle + \sum_{\xi \in I_0} \sum_{\xi' \notin I_0} a(\xi, \xi') \langle X_{\xi'}, f_\xi \rangle,$$

by Part 1 we have  $\langle b(X_s), \mathbf{f} \rangle \in \mathcal{H}$ .

Define  $\Phi_t(X) := \Phi(X_t)$ ,

$$\Lambda_n(\mathbf{f}, X) := \frac{2t}{n} \sum_{i=1}^n \langle b(X_{it/n}), \mathbf{f} \rangle$$

and

$$\tilde{\Phi}_t^{(n)}(X) := \Phi(X_t) e^{-\Lambda_n(\mathbf{f}, X)}.$$

Since both  $\Phi(X_t) \in \mathcal{H}$  and  $e^{-\Lambda_n(\mathbf{f}, X)} \in \mathcal{H}$  by Part 1, then  $\tilde{\Phi}_t^{(n)} \in \mathcal{H}$  and (3.11) holds with  $\Phi$  replaced by  $\tilde{\Phi}_t^{(n)}$ .

Clearly,

$$\text{bp} - \lim_{n \rightarrow \infty} \tilde{\Phi}_t^{(n)} = \tilde{\Phi}_t.$$

Similar to (3.10), we have

$$\frac{\delta \tilde{\Phi}_t^{(n)}(X)}{\delta X_\xi}(u) = \frac{\delta \Phi(X_t)}{\delta X_\xi}(u) e^{-\Lambda_n(\mathbf{f}, X)} - 2\tilde{\Phi}_t^{(n)}(X) \frac{t}{n} \sum_{i=1}^n \frac{\delta \langle b(X_{it/n}), \mathbf{f} \rangle}{\delta X_\xi}(u).$$

Therefore,

$$\text{bp} - \lim_{n \rightarrow \infty} \frac{\delta \tilde{\Phi}_t^{(n)}}{\delta X_\xi} = \frac{\delta \tilde{\Phi}_t}{\delta X_\xi}, \quad \forall \xi \in I,$$

and (3.11) holds for  $\tilde{\Phi}_t$ .

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