The maximum Markovian self-adjoint extensions of Dirichlet operators for interacting particle systems

Byron Schmuland and Wei Sun

Abstract. Let μ be a Ruelle measure on the configuration space $\Gamma_{\mathbb{R}^d}$ with a pair potential ϕ. Then the generator of the corresponding intrinsic Dirichlet form $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is an extension of the Dirichlet operator $\triangle_\phi^\Gamma$ on $L^2(\Gamma_{\mathbb{R}^d};\mu)$ with domain $\mathcal{F}C_b^\infty$ defined by $\triangle_\phi^\Gamma = \mathrm{div}_\phi^\Gamma \nabla^\Gamma$. Here $\mathcal{F}C_b^\infty$ is the set of smooth cylinder functions, ∇^Γ the gradient of the Riemannian structure on $\Gamma_{\mathbb{R}^d}$ and div_ϕ^Γ the corresponding divergence. For a large class of nonnegative (singular) potentials ϕ, we give a convergence characterization for the weak Sobolev spaces $W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ and prove that the generator of $(\mathcal{E}^\mu,\ W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is the maximum Markovian self-adjoint extension of $(\triangle_\phi^\Gamma, \mathcal{F}C_b^\infty)$. Furthermore, we construct stationary diffusion processes associated with $(\mathcal{E}^\mu,\ W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ by approximation.

1991 Mathematical Subject Classification: 60G60, 82C22; 31C25, 47D07.

Running Title: Markovian self-adjoint extensions of Dirichlet operators.

1 Introduction

In a pair [10, 16] of remarkable papers, Osada and Yoshida independently constructed infinite-dimensional diffusions with singular interactions through quasi-regular Dirichlet forms. In [1, 2], Albeverio, Kondratiev and Röckner laid the foundation for understanding such processes through their systematic study of analysis and geometry on configuration spaces. They proved that for a large class of pair potentials ϕ the corresponding canonical Gibbs measures μ on $\Gamma_{\mathbb{R}^d}$ can be characterized by an integration by parts formula. One consequence is that for such a μ the corresponding intrinsic Dirichlet form $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is a symmetric, quasi-regular and local Dirichlet form. It is associated with a conservative diffusion process on $\Gamma_{\mathbb{R}^d}$ with invariant measure μ. The corresponding generator is an extension of the Dirichlet operator $\triangle_\phi^\Gamma$ on $L^2(\Gamma_{\mathbb{R}^d};\mu)$ with domain $\mathcal{F}C_b^\infty$ defined by $\triangle_\phi^\Gamma := \mathrm{div}_\phi^\Gamma \nabla^\Gamma$. Here $\mathcal{F}C_b^\infty$ is the set of smooth cylinder functions, ∇^Γ the gradient of the Riemannian structure on $\Gamma_{\mathbb{R}^d}$ and div_ϕ^Γ the corresponding divergence.

We are concerned with the family

$$\mathcal{M}(\triangle_\phi^\Gamma) = \left\{ A : \begin{array}{l} A \text{ is a self–adjoint extension of } (\triangle_\phi^\Gamma, \mathcal{F}C_b^\infty) \text{ on } L^2(\Gamma_{\mathbb{R}^d};\mu) \\ \text{and the semigroup generated by } A \text{ is Markovian} \end{array} \right\}.$$

The family $\mathcal{M}(\triangle_\phi^\Gamma)$ is non-empty since the generator of $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$, which we denote by L_0^μ, is obviously in $\mathcal{M}(\triangle_\phi^\Gamma)$. For any $A \in \mathcal{M}(\triangle_\phi^\Gamma)$, the closed symmetric form $(\mathcal{E}_A, D(\mathcal{E}_A))$ corresponding to A satisfies $D(\mathcal{E}_A) \supset H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ and $\mathcal{E}_A(F,G) =$

$\mathcal{E}^\mu(F,G), F,G \in H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ (cf. [5, Lemma 3.3.1]). As usual we introduce a semi-order $\prec$ in $\mathcal{M}(\Delta_\phi^\Gamma)$ by

$$A_1 \prec A_2 \Longleftrightarrow \begin{array}{l} D(\mathcal{E}_{A_1}) \subset D(\mathcal{E}_{A_2}), \\ \mathcal{E}_{A_1}(F,F) \geq \mathcal{E}_{A_2}(F,F), \quad \forall F \in D(\mathcal{E}_{A_1}). \end{array}$$

Then, L_0^μ is the minimum element of $\mathcal{M}(\Delta_\phi^\Gamma)$ with the above semi-order, i.e., $L_0^\mu \prec A$, $\forall A \in \mathcal{M}(\Delta_\phi^\Gamma)$. In this article we concentrate on studying the maximum element of $\mathcal{M}(\Delta_\phi^\Gamma)$, i.e., the maximum Markovian self-adjoint extension of $(\Delta_\phi^\Gamma, \mathcal{F}C_b^\infty)$. We expect our results will shed light on the challenging Markov uniqueness problem, that is, whether there is exactly one element in $\mathcal{M}(\Delta_\phi^\Gamma)$. For more details about the Markov uniqueness problem in general framework we refer the interested reader to [4] and references therein.

The remainder of this article is organized as follows. In section 2, we recall some basic definitions and properties of Ruelle measures and intrinsic Dirichlet forms on $\Gamma_{\mathbb{R}^d}$. In Section 3, we analyze the differentiability of the system of density distributions and give a convergence characterization for the weak Sobolev space $W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$. Combining this characterization with the basic criterion in [4], we prove that if ϕ is nonnegative and $\nabla\phi$ decays polynomially at infinity, then the generator of $(\mathcal{E}^\mu,\ W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is the maximum Markovian self-adjoint extension of $(\Delta_\phi^\Gamma, \mathcal{F}C_b^\infty)$ (cf. Theorem 3.4 below). As an application, we consider the Dirichlet forms investigated by [3] and comment on some further properties of $W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$. In Section 4, we construct a stationary diffusion process associated with the maximum Dirichlet form $(\mathcal{E}^\mu,\ W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ based upon Mosco convergence, Lyons-Zheng decomposition and a uniqueness result of Dirichlet forms associated with systems of infinitely interacting particles living on a bounded domain in $\mathbb{R}^d$.

2 Ruelle measures and intrinsic Dirichlet forms on $\Gamma_{\mathbb{R}^d}$

In this section, we give some basic properties of Ruelle measures on configuration space. For more detailed definitions and fuller explanations we refer the reader to [15].

We define the space of locally finite configurations in $\mathbb{R}^d$ by

$$\Gamma_{\mathbb{R}^d} := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for every compact } K\}.$$

We identify a configuration γ with the Radon measure $\sum_{x\in\gamma}\varepsilon_x$ and give $\Gamma_{\mathbb{R}^d}$ the topology of vague convergence of measures. We define measures on $\Gamma_{\mathbb{R}^d}$ on the corresponding Borel sets $\mathcal{B}(\Gamma_{\mathbb{R}^d})$ and use $\mathcal{O}_c(\mathbb{R}^d)$, $\mathcal{B}_c(\mathbb{R}^d)$ to denote the family of bounded open sets, Borel sets in $\mathbb{R}^d$, respectively.

For $\gamma \in \Gamma_{\mathbb{R}^d}$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we define $\gamma_\Lambda = \gamma \cap \Lambda$. For $f \in C_0(\mathbb{R}^d)$, the continuous functions on $\mathbb{R}^d$ with compact support, we let $\langle f,\gamma\rangle$ be the integral of f with respect to the measure γ, that is, $\langle f,\gamma\rangle = \sum_{x\in\gamma} f(x)$. Let m be Lebesgue measure on $\mathbb{R}^d$.

The *Poisson measure* π_{zm} with intensity $z > 0$ is the probability measure on $\Gamma_{\mathbb{R}^d}$ characterized by:

$$\int_{\Gamma_{\mathbb{R}^d}} \exp(\langle f, \gamma \rangle)\, \pi_{zm}(d\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\, zm(dx) \right), \quad \forall f \in C_0(\mathbb{R}^d).$$

A *pair potential* is any measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x)$. Throughout this work, we suppose that the pair potential ϕ is superstable, lower regular, and integrable, and let μ be a tempered grand canonical Gibbs measure (Ruelle measure for short) with the pair potential ϕ, activity z, and inverse temperature β (cf. [15, 2, 11]). Then, μ satisfies the *Dobrushin-Lanford-Ruelle equations*: for any $F \in L^1(\Gamma_{\mathbb{R}^d}; \mu)$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$,

$$\int_{\Gamma_{\mathbb{R}^d}} F(\gamma)\mu(d\gamma) = \iint\limits_{\Gamma_{\mathbb{R}^d}\Gamma_{\mathbb{R}^d}} F(\gamma_{\Lambda^c} + \omega_\Lambda) \frac{\exp\left[-\beta E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda) \right]}{Z_\Lambda^{z,\phi}(\gamma)}\, \pi_{zm}(d\omega)\, \mu(d\gamma).$$

Here, for a configuration $\gamma \in \Gamma_{\mathbb{R}^d}$, $E_\Lambda^\phi(\gamma)$ denotes the *conditional energy* of γ in Λ:

$$E_\Lambda^\phi(\gamma) := \begin{cases} \sum\limits_{\substack{\{x,y\}\subset\gamma \\ \{x,y\}\cap\Lambda\neq\emptyset}} \phi(x-y), & \text{if} \sum\limits_{\substack{\{x,y\}\subset\gamma \\ \{x,y\}\cap\Lambda\neq\emptyset}} |\phi(x-y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$Z_\Lambda^{z,\phi}(\gamma) := \int_{\Gamma_{\mathbb{R}^d}} \exp\left[-\beta E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda) \right]\, \pi_{zm}(d\omega).$$

(We adopt the convention that a sum over the empty set is zero so that $E_\Lambda^\phi(\gamma) = 0$ if either $\gamma(\mathbb{R}^d) = 1$ or $\gamma(\Lambda) = 0$.)

For a pair potential ϕ, we define the *energy* of n particles $\{x_1, \ldots, x_n\}$ by the formula

$$U(x_1, \ldots, x_n) := \begin{cases} \sum\limits_{i<j} \phi(x_i - x_j), & \text{if } n \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

For $\triangle \in \mathcal{B}_c(\mathbb{R}^d)$, we define

$$E_\triangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\triangle^n} e^{-\beta U(x_1,\ldots,x_n)} m(dx_1) \cdots m(dx_n)$$

to be the grand partition function. For $\Lambda,\ \triangle \in \mathcal{B}_c(\mathbb{R}^d)$, we define

$$(1) \qquad \rho_{\triangle\Lambda}^n(x_1, \ldots, x_n) = E_\triangle^{-1} \sum_{k=n}^{\infty} \frac{z^k}{(k-n)!} \int_{(\triangle\setminus\Lambda)^{k-n}} e^{-\beta U(x_1,\ldots,x_k)} m(dx_{n+1}) \cdots m(dx_k)$$

if $x_1, \dots, x_n \in \Lambda$, $= 0$ otherwise. By [15, Proposition 2.6], there exists a constant $\xi > 0$ such that for all $\Lambda, \Delta \in \mathcal{B}_c(\mathbb{R}^d)$, $x_1, \dots, x_n \in \Lambda$,

$$\rho^n_{\Delta\Lambda}(x_1, \dots, x_n) \le \xi^n, \quad \forall n \in \mathbb{Z}_+. \tag{2}$$

By [15, Theorem 5.5], from every sequence $\{\Delta_l \in \mathcal{B}_c(\mathbb{R}^d)\}$ tending to $\mathbb{R}^d$ one can extract a subsequence $\{\Delta_{l'}\}$ such that (for each $n \in \mathbb{Z}_+$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$) the following limit exists uniformly in $x_1, \dots, x_n$

$$\lim_{l' \to \infty} \rho^n_{\Delta_{l'}\Lambda}(x_1, \dots, x_n) = \sigma^n_\Lambda(x_1, \dots, x_n).$$

We call $\{\sigma^n_\Lambda\}$ the *system of density distributions*, which are positive measurable functions on Λ^n invariant with respect to permutations of the coordinates $x_1, \dots, x_n$. They satisfy the following conditions:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sigma^n_\Lambda(x_1, \dots, x_n) m(dx_1) \cdots m(dx_n) = 1$$

and if $\Lambda \subset \Delta$

$$\sigma^n_\Lambda(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{(\Delta \setminus \Lambda)^k} \sigma^{n+k}_\Delta(x_1, \dots, x_{n+k}) m(dx_{n+1}) \cdots m(dx_{n+k}).$$

For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we define $\Gamma_\Lambda := \{\gamma \in \Gamma_{\mathbb{R}^d} : \gamma(\mathbb{R}^d \setminus \Lambda) = 0\}$, and for $n \in \mathbb{Z}_+$ we define $\Gamma^{(n)}_\Lambda := \{\gamma \in \Gamma_\Lambda : \gamma(\Lambda) = n\}$. Then

$$\Gamma_\Lambda = \bigcup_{n=0}^{\infty} \Gamma^{(n)}_\Lambda. \tag{3}$$

We define $p_\Lambda : \Gamma_{\mathbb{R}^d} \to \Gamma_\Lambda$ by $p_\Lambda(\gamma) := \gamma_\Lambda$ and denote $\mu_\Lambda = \mu \circ p_\Lambda^{-1}$. Corresponding to the decomposition (3), there is an induced decomposition of $\mu_\Lambda : \mu_\Lambda = \sum_{n=0}^{\infty} \mu^{(n)}_\Lambda$. We denote $\tilde{\Lambda}^n := \{(x_1, \dots, x_n) \in \Lambda^n : x_i \ne x_j, \ \forall i \ne j\}$ and define $s^n_\Lambda : \tilde{\Lambda}^n \to \Gamma^{(n)}_\Lambda$ by $s^n_\Lambda((x_1, \dots, x_n)) := \sum_{i=1}^n \varepsilon_{x_i}$. Let $m_{\Lambda,n} := m^{\otimes n} \circ (s^n_\Lambda)^{-1}$. Then, we have the representation

$$\mu^{(n)}_\Lambda(d\gamma_n) = \frac{1}{n!} \sigma^n_\Lambda((s^n_\Lambda)^{-1}\gamma_n) m_{\Lambda,n}(d\gamma_n), \qquad \gamma_n \in \Gamma^{(n)}_\Lambda, n \in \mathbb{Z}_+.$$

In the remainder of this section we give some of the preliminaries for intrinsic Dirichlet forms on configuration space. For more details we refer the reader to [2, 11, 8]. We define

$$\mathcal{F}C_b^\infty := \{F : F(\gamma) = g(\langle f_1, \gamma \rangle, \dots, \langle f_n, \gamma \rangle) \\ \text{for some } f_i \in C_0^\infty(\mathbb{R}^d) \text{ and } g \in C_b^\infty(\mathbb{R}^n)\}.$$

For $F \in \mathcal{F}C_b^\infty$, we define the gradient $\nabla^\Gamma F$ at the point $\gamma \in \Gamma_{\mathbb{R}^d}$ as an element of the "tangent space" $T_\gamma\Gamma_{\mathbb{R}^d} := L^2(\mathbb{R}^d \to \mathbb{R}^d; \gamma)$ by the formula

$$(\nabla^\Gamma F)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i} \left(\langle f_1, \gamma\rangle, \ldots, \langle f_n, \gamma\rangle\right) \nabla f_i(x).$$

Here ∇ refers to the usual gradient on $\mathbb{R}^d$.

Until the end of this section we suppose that the pair potential ϕ satisfies the following conditions:

(D) $e^{-\beta\phi}$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d\backslash\{0\}$ and the weak gradient $\nabla\phi$ (which is a locally m-integrable function on $\mathbb{R}^d\backslash\{0\}$) considered as an m-a.e. defined function on $\mathbb{R}^d$ satisfies

$$\nabla\phi \in L^1(\mathbb{R}^d; e^{-\beta\phi}m) \cap L^2(\mathbb{R}^d; e^{-\beta\phi}m).$$

(P) There exist constants $c, r > 0$ and $v > d$ such that $|\nabla\phi(x)| \leq c(1 + |x|)^{-v}$ for $|x| > r$.

Note that many typical potentials in Statistical Physics (e.g. Lennard-Jones Potential) satisfy (D) and (P).

Definition 2.1. For $F, G \in \mathcal{F}C_b^\infty$ we define the *intrinsic pre-Dirichlet form* by

$$\begin{aligned}\mathcal{E}^\mu(F, G) :=& \int_{\Gamma_{\mathbb{R}^d}} \langle \nabla^\Gamma F, \nabla^\Gamma G\rangle_{T_\gamma(\Gamma_{\mathbb{R}^d})} \mu(d\gamma) \\ =& \int_{\Gamma_{\mathbb{R}^d}} \int_{\mathbb{R}^d} \langle (\nabla^\Gamma F)(\gamma; x), (\nabla^\Gamma G)(\gamma; x)\rangle_{\mathbb{R}^d}\, \gamma(dx)\mu(d\gamma).\end{aligned}$$

For any section $h : \Gamma_{\mathbb{R}^d} \to T\Gamma_{\mathbb{R}^d}$ (i.e., h satisfies $h(\gamma) \in T_\gamma\Gamma_{\mathbb{R}^d}$ for each $\gamma \in \Gamma_{\mathbb{R}^d}$), we define for $\varepsilon = -, +$

$$|h|_\varepsilon^2(\gamma) = \sum_{x\in\gamma} (1 + |x|)^{\varepsilon v} |h(\gamma, x)|_{\mathbb{R}^d}^2 \quad \text{and} \quad ||h||_\varepsilon^2 = \int_{\Gamma_{\mathbb{R}^d}} |h|_\varepsilon^2(\gamma)\mu(d\gamma).$$

We denote

$$b(\gamma) := \left(-2\beta \sum_{y\in\gamma\backslash x} \nabla\phi(x - y)\right)_{x\in\gamma}.$$

Then, as in [3, Theorem II.11, P.6521-6523], one finds that $||b||_- < \infty$ by condition (P).

Let $V_0(\mathbb{R}^d)$ denote the set of smooth vector fields on $\mathbb{R}^d$ with compact support. We identify each $v \in V_0(\mathbb{R}^d)$ with the constant vector field $\gamma \to v$ in $T\Gamma_{\mathbb{R}^d}$ and define

$$L_v^\phi := \langle b, v\rangle_{T\Gamma_{\mathbb{R}^d}} \quad \text{and} \quad B_v^\phi := L_v^\phi + \langle \operatorname{div} v, \cdot\rangle.$$

For $v \in V_0(\mathbb{R}^d)$ and $F \in \mathcal{F}C_b^\infty$ we define $\nabla_v^\Gamma F := \langle \nabla^\Gamma F, v \rangle_{T\Gamma_{\mathbb{R}^d}}$. Let $\mathcal{V}\mathcal{F}C_b^\infty$ be the set of all maps defined as follows:

$$\Gamma_{\mathbb{R}^d} \ni \gamma \to \sum_{i=1}^N F_i(\gamma) v_i,$$

where $F_1, \ldots, F_N \in \mathcal{F}C_b^\infty$; $v_1, \ldots, v_N \in V_0(\mathbb{R}^d)$. For $V = \sum_{i=1}^N F_i v_i \in \mathcal{V}\mathcal{F}C_b^\infty$ we define

$$\mathrm{div}_\phi^\Gamma V := \sum_{i=1}^N (\nabla_{v_i}^\Gamma F_i + B_{v_i}^\phi F_i).$$

For all $v \in V_0(\mathbb{R}^d)$ and $F, G \in \mathcal{F}C_b^\infty$, following [2, Theorem 4.3], the integration by parts formula holds:

$$\int_{\Gamma_{\mathbb{R}^d}} \nabla_v^\Gamma F G d\mu = -\int_{\Gamma_{\mathbb{R}^d}} F \nabla_v^\Gamma G d\mu - \int_{\Gamma_{\mathbb{R}^d}} F G B_v^\phi d\mu. \tag{4}$$

Or equivalently for all $F \in \mathcal{F}C_b^\infty, V \in \mathcal{V}\mathcal{F}C_b^\infty$,

$$\int_{\Gamma_{\mathbb{R}^d}} \langle \nabla^\Gamma F, V \rangle_{T\Gamma_{\mathbb{R}^d}} d\mu = -\int_{\Gamma_{\mathbb{R}^d}} F \mathrm{div}_\phi^\Gamma V d\mu.$$

The form $(\mathcal{E}^\mu, \mathcal{F}C_b^\infty)$ is then closable and its closure $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$ is a symmetric Dirichlet form on $L^2(\Gamma_{\mathbb{R}^d}; \mu)$. Moreover, $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$ is a local Dirichlet form. In order to associate it with a diffusion process, it is necessary to use the completed state space

$$\overline{\Gamma}_{\mathbb{R}^d} := \{\mathbb{Z}_+ \cup \{+\infty\}\text{-valued Radon measures on } \mathbb{R}^d\}.$$

Since $\Gamma_{\mathbb{R}^d} \subset \overline{\Gamma}_{\mathbb{R}^d}$ and $\mathcal{B}(\overline{\Gamma}_{\mathbb{R}^d}) \cap \Gamma_{\mathbb{R}^d} = \mathcal{B}(\Gamma_{\mathbb{R}^d})$, we can consider μ as a measure on $(\overline{\Gamma}_{\mathbb{R}^d}, \mathcal{B}(\overline{\Gamma}_{\mathbb{R}^d}))$ and correspondingly $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$ as a Dirichlet form on $L^2(\overline{\Gamma}_{\mathbb{R}^d}; \mu)$. On this completed space $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$ is quasi-regular, so there exists a conservative diffusion process associated with $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$ ([2, Theorem 5.2]). Note that by [13], the set $\overline{\Gamma}_{\mathbb{R}^d} \setminus \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}^\mu$-exceptional if $d \geq 2$.

Before ending this section let us briefly recall the construction of the weak Sobolev space $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$. Let $((\mathrm{div}_\phi^\Gamma)^*, D(\mathrm{div}_\phi^\Gamma)^*)$ be the adjoint of $(\mathrm{div}_\phi^\Gamma, \mathcal{V}\mathcal{F}C_b^\infty)$ as an operator from $L^2(\Gamma_{\mathbb{R}^d} \to T\Gamma_{\mathbb{R}^d}; \mu)$ to $L^2(\Gamma_{\mathbb{R}^d}; \mu)$. By definition, $G \in L^2(\Gamma_{\mathbb{R}^d}; \mu)$ belongs to $D(\mathrm{div}_\phi^\Gamma)^*$ if and only if there exists (unique) $(\mathrm{div}_\phi^\Gamma)^* G \in L^2(\Gamma_{\mathbb{R}^d} \to T\Gamma_{\mathbb{R}^d}; \mu)$ such that

$$\int_{\Gamma_{\mathbb{R}^d}} G \mathrm{div}_\phi^\Gamma V d\mu = -\int_{\Gamma_{\mathbb{R}^d}} \langle (\mathrm{div}_\phi^\Gamma)^* G, V \rangle_{T\Gamma_{\mathbb{R}^d}} d\mu, \quad \forall V \in \mathcal{V}\mathcal{F}C_b^\infty.$$

We set

$$W^{1,2}(\Gamma_{\mathbb{R}^d}; \mu) := D((\mathrm{div}_\phi^\Gamma)^*), \quad d^\mu := (\mathrm{div}_\phi^\Gamma)^*$$

and think of $W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ as a *weak (1,2)-Sobolev space* on $\Gamma_{\mathbb{R}^d}$ with norm

$$W^{1,2}(\Gamma_{\mathbb{R}^d};\mu) \ni G \to$$
$$||G||_{W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)} := \left(\int_{\Gamma_{\mathbb{R}^d}} \langle d^\mu G, d^\mu G \rangle_{T\Gamma_{\mathbb{R}^d}} d\mu + \int_{\Gamma_{\mathbb{R}^d}} G^2 d\mu \right)^{1/2}.$$

By (4), it follows that

$$\mathcal{F}C_b^\infty \subset W^{1,2}(\Gamma_{\mathbb{R}^d};\mu) \text{ and } d^\mu = \nabla^\Gamma \text{ on } \mathcal{F}C_b^\infty,$$

hence the densely defined positive definite symmetric bilinear form

$$(F,G) \to \int_{\Gamma_{\mathbb{R}^d}} \langle d^\mu F, d^\mu G \rangle_{T\Gamma_{\mathbb{R}^d}} d\mu$$

with domain $W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ extends $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ and is, therefore, denoted by $(\mathcal{E}^\mu,$
$W^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$. We use $W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ to denote the closure of $W^{1,2}(\Gamma_{\mathbb{R}^d};\mu) \cap L^\infty(\Gamma_{\mathbb{R}^d};\mu)$ in $W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ with respect to the $||\cdot||_{W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)}$-norm.

3 A convergence characterization for $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$

For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we let $\mathcal{F}(\Lambda)$ be the σ-algebra of events $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$ that only depend on the part of the configuration in Λ, that is, $1_\Delta(\gamma) = 1_\Delta(\gamma_\Lambda)$ for every $\gamma \in \Gamma_{\mathbb{R}^d}$. For an $\mathcal{F}(\Lambda)$-measurable function F, we have the unique decomposition

$$F(\gamma) = \sum_{n=0}^\infty F_\Lambda^n((s_\Lambda^n)^{-1}\gamma_\Lambda),$$

where F_Λ^n is a symmetric function on $\tilde{\Lambda}^n$ for $n \in \mathbb{Z}_+$.

Proposition 3.1. Suppose that a superstable, lower regular and integrable pair potential ϕ is nonnegative and satisfies condition (D). Then, for each $n \in \mathbb{N}$, $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, $\sigma_\Lambda^n \in H^{1,1}(\tilde{\Lambda}^n; m^{\otimes n})$ and

$$\sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} \sum_{i=1}^n |\nabla_{x_i} \sigma_\Lambda^n(x_1,\ldots,x_n)|_{\mathbb{R}^d} m(dx_1)\cdots m(dx_n) < \infty.$$

Moreover, $\sqrt{\sigma_\Lambda^n} \in H^{1,2}(\tilde{\Lambda}^n; m^{\otimes n})$ and

$$\sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} \sum_{i=1}^n |\nabla_{x_i} \sqrt{\sigma_\Lambda^n}(x_1,\ldots,x_n)|_{\mathbb{R}^d}^2 m(dx_1)\cdots m(dx_n) < \infty.$$

Proof. Let $\triangle \in \mathcal{B}_c(\mathbb{R}^d)$ with $\Lambda \subset \triangle$. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \Lambda$, let us consider the term

$$J^n_\triangle = E^{-1}_\triangle \sum_{k=n+1}^{\infty} \frac{z^k}{(k-n)!} \int_{(\triangle\setminus\Lambda)^{k-n}} \sum_{j=n+1}^{k} \nabla\phi(x_i - x_j)\, e^{-\beta U(x_1,\ldots,x_k)}\, m(dx_{n+1})\cdots m(dx_k).$$

By the symmetry in the integrand, the inside sum contributes $k-n$ like terms, so we may rewrite it as

$$J^n_\triangle = E^{-1}_\triangle \sum_{k=n+1}^{\infty} \frac{z^k}{(k-n-1)!} \int_{(\triangle\setminus\Lambda)^{k-n}} \nabla\phi(x_i - x_{n+1})\, e^{-\beta U(x_1,\ldots,x_k)}\, m(dx_{n+1})\cdots m(dx_k).$$

Let us now separate the integral over the variable x_{n+1} from the rest. In this way we obtain

$$J^n_\triangle = E^{-1}_\triangle \sum_{k=n+1}^{\infty} \frac{z^k}{(k-n-1)!} \int_{(\triangle\setminus\Lambda)^{k-n-1}} I\, e^{-\beta U(x_1,\ldots,x'_{n+1},\ldots,x_k)}\, m(dx_{n+2})\cdots m(dx_k),$$

where

$$I = \int_{\triangle\setminus\Lambda} \nabla\phi(x_i - x_{n+1})\, e^{-\beta\sum_{j=1,j\neq n+1}^{k} \phi(x_j - x_{n+1})}\, m(dx_{n+1})$$

and $(x_1, \ldots, x'_{n+1}, \ldots, x_k)$ is the vector of length $k-1$ obtained from $(x_1, \ldots, x_k)$ by removing the entry x_{n+1}. Since the potential function ϕ is nonnegative, if we bound the exponential in I by keeping only the term $\phi(x_i - x_{n+1})$, we get

$$\begin{aligned} |I| &\leq \int_{\triangle\setminus\Lambda} |\nabla\phi(x_i - x_{n+1})|\, e^{-\beta\phi(x_i - x_{n+1})}\, m(dx_{n+1}) \\ &\leq \int_{\mathbb{R}^d} |\nabla\phi(x)|\, e^{-\beta\phi(x)}\, m(dx) = c_1 < \infty. \end{aligned}$$

Applying this bound to $J^n_\triangle$ we obtain

$$\begin{aligned} |J^n_\triangle| &\leq c_1\, E^{-1}_\triangle \sum_{k=n+1}^{\infty} \frac{z^k}{(k-n-1)!} \\ &\quad \cdot \int_{(\triangle\setminus\Lambda)^{k-n-1}} e^{-\beta U(x_1,\ldots,x'_{n+1},\ldots,x_k)}\, m(dx_{n+2})\cdots m(dx_k) \\ &= c_1\, z\, \rho^n_{\triangle\Lambda}. \end{aligned} \tag{5}$$

Then, we may take an increasing sequence $\{\triangle_l : \Lambda \subset \triangle_l \in \mathcal{B}_c(\mathbb{R}^d)\}$ tending to $\mathbb{R}^d$ such that $\{J^n_{\triangle_l}\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\tilde{\Lambda}^n; m^{\otimes n})$ with respect to the weak topology of $L^2(\tilde{\Lambda}^n; m^{\otimes n})$. We assume without loss of generality that the following limit exists uniformly in $x_1, \ldots, x_n \in \Lambda$

$$\lim_{l\to\infty} \rho^n_{\triangle_l\Lambda}(x_1, \ldots, x_n) = \sigma^n_\Lambda(x_1, \ldots, x_n). \tag{6}$$

Note that for $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \Lambda$,

$$\nabla_{x_i} \rho^n_{\Delta\Lambda}(x_1, \ldots, x_n) = \beta \left(\sum_{j \neq i}^{n} \nabla \phi(x_i - x_j) \left(\rho^n_{\Delta\Lambda}(x_1, \ldots, x_n) - \frac{z^n}{E_\Delta} \right) + J^n_\Delta(x_1, \ldots, x_n) \right).$$

For every compact subset K of $\tilde{\Lambda}^n$, we denote by χ_K the characteristic function of K. Then, $\{\nabla_{x_i} \rho^n_{\Delta_l \Lambda} \chi_K\}_{l \in \mathbb{N}}$ is a Cauchy sequence in $L^1(\tilde{\Lambda}^n; m^{\otimes n})$ with respect to the weak topology of $L^1(\tilde{\Lambda}^n; m^{\otimes n})$. Thus, by (2), (5), and the dominated convergence theorem, we conclude that $\sigma^n_\Lambda \in H^{1,1}_{\mathrm{loc}}(\tilde{\Lambda}^n; m^{\otimes n})$ and

$$\sum_{i=1}^{n} |\nabla_{x_i} \sigma^n_\Lambda(x_1, \ldots, x_n)|_{\mathbb{R}^d} \leq \beta \sigma^n_\Lambda(x_1, \ldots, x_n) \left[\sum_{i=1}^{n} \sum_{j \neq i}^{n} |\nabla \phi(x_i - x_j)| + c_1 n z \right].$$

Following the argument similar to [3, Theorem II.11, P.6522-6523], we obtain from (D) and [3, Lemma III.2] that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sum_{i=1}^{n} |\nabla_{x_i} \sigma^n_\Lambda(x_1, \ldots, x_n)|_{\mathbb{R}^d} m(dx_1) \cdots m(dx_n) \\
&\quad \leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \beta \sigma^n_\Lambda(x_1, \ldots, x_n) \left[\sum_{i=1}^{n} \sum_{j \neq i}^{n} |\nabla \phi(x_i - x_j)| + c_1 n z \right] m(dx_1) \cdots m(dx_n) \\
&\quad = \beta \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} n \sum_{j=2}^{n} |\nabla \phi(x_1 - x_j)| \sigma^n_\Lambda(x_1, \ldots, x_n) m(dx_1) \cdots m(dx_n) + c_2 \\
&\quad < \infty.
\end{aligned}$$

By (6), (2), and (5), it is easy to see that

$$\nabla_{x_i} \sqrt{\sigma^n_\Lambda(x_1, \ldots, x_n)} = \begin{cases} \frac{\nabla_{x_i} \sigma^n_\Lambda(x_1, \ldots, x_n)}{2\sqrt{\sigma^n_\Lambda(x_1, \ldots, x_n)}}, & \text{if } \sigma^n_\Lambda(x_1, \ldots, x_n) \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and $\sqrt{\sigma^n_\Lambda} \in H^{1,1}_{\mathrm{loc}}(\tilde{\Lambda}^n; m^{\otimes n})$. Moreover,

$$\sum_{i=1}^{n} |\nabla_{x_i} \sqrt{\sigma^n_\Lambda}(x_1, \ldots, x_n)|^2_{\mathbb{R}^d} \leq \frac{\beta^2 \sigma^n_\Lambda(x_1, \ldots, x_n)}{2} \left\{ \sum_{i=1}^{n} \left[\sum_{j \neq i}^{n} |\nabla \phi(x_i - x_j)| \right]^2 + n c_1^2 z^2 \right\}.$$

Following the argument similar to [3, Theorem II.11, P.6522-6523] again, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sum_{i=1}^{n} |\nabla_{x_i} \sqrt{\sigma_\Lambda^n}(x_1, \ldots, x_n)|_{\mathbb{R}^d}^2 m(dx_1) \cdots m(dx_n) \\
&\quad \leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \frac{\beta^2 \sigma_\Lambda^n(x_1, \ldots, x_n)}{2} \left\{ \sum_{i=1}^{n} \left[\sum_{j \neq i}^{n} |\nabla \phi(x_i - x_j)| \right]^2 + n c_1^2 z^2 \right\} \\
&\quad\quad m(dx_1) \cdots m(dx_n) \\
&\quad = \frac{\beta^2}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} n \left[\sum_{j=2}^{n} |\nabla \phi(x_1 - x_j)| \right]^2 \sigma_\Lambda^n(x_1, \ldots, x_n) m(dx_1) \cdots m(dx_n) + c_3 \\
&\quad < \infty.
\end{aligned}$$

This completes the proof.

For $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, we let $H(\Lambda; \mu)$ denote the set of all bounded $\mathcal{F}(\Lambda)$-measurable functions F such that $F_\Lambda^n \in H_{\text{loc}}^{1,1}(\tilde{\Lambda}^n; m^{\otimes n})$ for each $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sum_{i=1}^{n} |\nabla_{x_i} F_\Lambda^n(x_1, \ldots, x_n)|_{\mathbb{R}^d}^2 \sigma_\Lambda^n(x_1, \ldots, x_n) m(dx_1) \cdots m(dx_n) < \infty.$$

For $F \in H(\Lambda; \mu)$ we define

$$D_\Lambda^\mu F(\gamma; x) := \begin{cases} \nabla_{x_i} F_\Lambda^n(x_1, \ldots, x_n), \ x = x_i, 1 \leq i \leq n, \\ 0, \ x \notin \gamma_\Lambda \end{cases}$$

if $\gamma_\Lambda = \{x_i : 1 \leq i \leq n\}$ for some $n \in \mathbb{Z}_+$. Then, $D_\Lambda^\mu F \in L^2(\Gamma_{\mathbb{R}^d} \to T\Gamma_{\mathbb{R}^d}; \mu)$. We denote $\varphi_\Lambda(\gamma) := \sum_{n=0}^{\infty} \sqrt{\sigma_\Lambda^n}((s_\Lambda^n)^{-1}\gamma_\Lambda)$. With slight abuse of notation, we define

$$D_\Lambda^\mu \varphi_\Lambda^2(\gamma) := \begin{cases} \nabla_{x_i} \sigma_\Lambda^n(x_1, \ldots, x_n), \ x = x_i, 1 \leq i \leq n, \\ 0, \ x \notin \gamma_\Lambda \end{cases}$$

if $\gamma_\Lambda = \{x_i : 1 \leq i \leq n\}$ for some $n \in \mathbb{Z}_+$.

Let $\mathbb{H}^\mu$ denote the set of all $F \in L^\infty(\Gamma_{\mathbb{R}^d}; \mu)$ such that there exist an increasing sequence $\{\Delta_l \in \mathcal{O}_c(\mathbb{R}^d)\}$ tending to $\mathbb{R}^d$ and a sequence of functions $\{F_l \in H(\Delta_l; \mu)\}$ satisfying $F_l \to F$ in $L^2(\Gamma_{\mathbb{R}^d}; \mu)$ as $l \to \infty$, and there exists an element in $L^2(\Gamma_{\mathbb{R}^d} \to T\Gamma_{\mathbb{R}^d}; \mu)$, which we denote by $D^\mu F$, such that

$$\lim_{l \to \infty} \int_{\Gamma_{\mathbb{R}^d}} |\langle D_{\Delta_l}^\mu F_l - D^\mu F, V \rangle_{T\Gamma_{\mathbb{R}^d}}| d\mu = 0, \quad \forall V \in \mathcal{VFC}_b^\infty. \tag{7}$$

If (7) is satisfied, we denote $D_{\Delta_l}^\mu F_l \xrightarrow{w} D^\mu F$ and define

$$\mathcal{E}^{\mathbb{H}^\mu}(F, F) := \int_{\Gamma_{\mathbb{R}^d}} \langle D^\mu F, D^\mu F \rangle_{T\Gamma_{\mathbb{R}^d}} d\mu. \tag{8}$$

Of course, one must check that (8) is well-defined, i.e., independent of the particular choices of $\{\triangle_l\}$ and $\{F_l\}$. This will become clear after we prove Theorem 3.3 below.

In the sequel, for $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, we let $V_0(\Lambda)$ denote the set of smooth vector fields on $\mathbb{R}^d$ with support in Λ. We define

$$\mathcal{F}C_b^\infty(\Lambda) := \{F : F(\gamma) = g(\langle f_1, \gamma\rangle, \dots, \langle f_n, \gamma\rangle) \quad \text{for some } f_i \in C_0^\infty(\Lambda) \text{ and } g \in C_b^\infty(\mathbb{R}^n)\}.$$

We use $\{e_j\}_{1\le j\le d}$ to denote the standard bases of $\mathbb{R}^d$.

Lemma 3.2. For any $v \in V_0(\Lambda)$,

$$\langle D^\mu_\Lambda \varphi^2_\Lambda, v\rangle_{T\Gamma_{\mathbb{R}^d}} = E_\mu[L^\phi_v | \mathcal{F}(\Lambda)]\varphi^2_\Lambda, \quad \mu - a.e. \tag{9}$$

Proof. By Proposition 3.1, $\sigma^n_\Lambda \in H^{1,1}(\tilde{\Lambda}^n; m^{\otimes n})$ for $n \in \mathbb{N}$. Note that σ^n_Λ is invariant with respect to permutations of the coordinates $x_1, \dots, x_n$, we then conclude from integration by parts and approximation that for each $n \in \mathbb{N}$ and any $F \in C_0^\infty(\Lambda^n)$ which is invariant with respect to permutations of the coordinates $x_1, \dots, x_n$,

$$\int_{\Lambda^n} \nabla_{x_i}(F\sigma^n_\Lambda)(x_1, \dots, x_n) m(dx_1)\cdots m(dx_n) = 0, \quad \forall 1 \le i \le n. \tag{10}$$

We fix a $v = \sum_{j=1}^d v_j e_j \in V_0(\Lambda)$ and write $E_\mu[L^\phi_v|\mathcal{F}(\Lambda)] = L^\phi_{v\Lambda} \circ p_\Lambda$, $\mu - a.e.$ Let $G \in \mathcal{F}C_b^\infty(\Lambda)$. Since

$$\begin{aligned}
0 =& \int_{\Gamma_{\mathbb{R}^d}} \mathrm{div}^\Gamma_\phi(Gv)d\mu \\
=& \sum_{n=0}^\infty \frac{1}{n!}\int_{\Lambda^n}\Bigg\{\sum_{i=1}^n \Bigg[\Bigg\langle \nabla_{x_i} G\left(\sum_{i=1}^n \varepsilon_{x_i}\right), v(x_i)\Bigg\rangle_{\mathbb{R}^d} \sigma^n_\Lambda(x_1, \dots, x_n) \\
&+ G\left(\sum_{i=1}^n \varepsilon_{x_i}\right)\sum_{j=1}^d \partial_j v_j(x_i)\sigma^n_\Lambda(x_1, \dots, x_n)\Bigg] \\
&+ (GL^\phi_{v\Lambda})\left(\sum_{i=1}^n \varepsilon_{x_i}\right)\sigma^n_\Lambda(x_1, \dots, x_n)\Bigg\} m(dx_1)\cdots m(dx_n),
\end{aligned}$$

we get

$$\begin{aligned}
&\sum_{n=0}^\infty \frac{1}{n!}\int_{\Lambda^n}(GL^\phi_{v\Lambda})\left(\sum_{i=1}^n \varepsilon_{x_i}\right)\sigma^n_\Lambda(x_1, \dots, x_n) m(dx_1)\cdots m(dx_n) \\
&= -\sum_{n=0}^\infty \frac{1}{n!}\int_{\Lambda^n}\sum_{i=1}^n \Bigg[\Bigg\langle \nabla_{x_i} G\left(\sum_{i=1}^n \varepsilon_{x_i}\right), v(x_i)\Bigg\rangle_{\mathbb{R}^d}\sigma^n_\Lambda(x_1, \dots, x_n) \\
&\quad + G\left(\sum_{i=1}^n \varepsilon_{x_i}\right)\sum_{j=1}^d \partial_j v_j(x_i)\sigma^n_\Lambda(x_1, \dots, x_n)\Bigg] m(dx_1)\cdots m(dx_n).
\end{aligned} \tag{11}$$

Substituting the F in (10) by $Gv_j,\ 1\le j\le d$, respectively, and adding them up, we get

$$0=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sum_{i=1}^{n}\Bigg[\left\langle\nabla_{x_i}G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right),v(x_i)\right\rangle_{\mathbb{R}^d}\sigma_\Lambda^n(x_1,\ldots,x_n)$$
$$+G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\sum_{j=1}^{d}\partial_j v_j(x_i)\sigma_\Lambda^n(x_1,\ldots,x_n)$$
$$+G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\langle\nabla_{x_i}\sigma_\Lambda^n(x_1,\ldots,x_n),v(x_i)\rangle_{\mathbb{R}^d}\Bigg]m(dx_1)\cdots m(dx_n).$$

Then

$$\text{(12)}\quad \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sum_{i=1}^{n}G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\langle\nabla_{x_i}\sigma_\Lambda^n(x_1,\ldots,x_n),v(x_i)\rangle_{\mathbb{R}^d}m(dx_1)\cdots m(dx_n)$$
$$=-\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sum_{i=1}^{n}\Bigg[\left\langle\nabla_{x_i}G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right),v(x_i)\right\rangle_{\mathbb{R}^d}\sigma_\Lambda^n(x_1,\ldots,x_n)$$
$$+G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\sum_{j=1}^{d}\partial_j v_j(x_i)\sigma_\Lambda^n(x_1,\ldots,x_n)\Bigg]m(dx_1)\cdots m(dx_n).$$

By (11) and (12), we get

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sum_{i=1}^{n}G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\langle\nabla_{x_i}\sigma_\Lambda^n(x_1,\ldots,x_n),v(x_i)\rangle_{\mathbb{R}^d}m(dx_1)\cdots m(dx_n)$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}(GL^\phi_{v\Lambda})\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\sigma_\Lambda^n(x_1,\ldots,x_n)m(dx_1)\cdots m(dx_n),$$

i.e.,

$$\int_{\Gamma_{\mathbb{R}^d}}G\frac{\langle D_\Lambda^\mu\varphi_\Lambda^2,v\rangle_{T\Gamma_{\mathbb{R}^d}}}{\varphi_\Lambda^2}d\mu=\int_{\Gamma_{\mathbb{R}^d}}GE_\mu[L_v^\phi|\mathcal{F}(\Lambda)]d\mu.$$

By a monotone class argument, it is easy to see that $\mathcal{F}C_b^\infty(\Lambda)$ is dense in $L^2(\Gamma_\Lambda;\mu_\Lambda)$ and therefore (9) holds.

For $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$, we let $S(\Lambda)$ denote the closed subspace of $L^2(\Gamma_{\mathbb{R}^d}\to T\Gamma_{\mathbb{R}^d};\mu)$ generated by $\mathcal{VF}C_b^\infty(\Lambda)$, where $\mathcal{VF}C_b^\infty(\Lambda)$ is the set of all maps: $\Gamma_{\mathbb{R}^d}\ni\gamma\to\sum_{i=1}^N F_i(\gamma)v_i$ such that $F_1,\ldots,F_N\in\mathcal{F}C_b^\infty(\Lambda)$ and $v_1,\ldots,v_N\in V_0(\Lambda)$. For $U\in L^2(\Gamma_{\mathbb{R}^d}\to T\Gamma_{\mathbb{R}^d};\mu)$, we denote by $E_\mu[U|\mathcal{F}(\Lambda)]$ the unique projection of U on the subspace $S(\Lambda)$. More generally, for any section $U:\Gamma_{\mathbb{R}^d}\to T\Gamma_{\mathbb{R}^d}$ satisfying $||U||_-<\infty$, we define $E_\mu[U|\mathcal{F}(\Lambda)]\in$

$S(\Lambda)$ by:

$$\int_{\Gamma_{\mathbb{R}^d}} \langle E_\mu[U|\mathcal{F}(\Lambda)], V\rangle_{T\Gamma_{\mathbb{R}^d}} d\mu = \int_{\Gamma_{\mathbb{R}^d}} \langle U, V\rangle_{T\Gamma_{\mathbb{R}^d}} d\mu, \quad \forall V \in \mathcal{V}\mathcal{F}C_b^\infty(\Lambda).$$

Theorem 3.3. Suppose that a superstable, lower regular and integrable pair potential ϕ is nonnegative and satisfies conditions (D) and (P). Then $(\mathcal{E}^{\mathbb{H}^\mu}, \mathbb{H}^\mu)$ is closable and its closure $(\mathcal{E}^{\mathbb{H}^\mu}, \mathcal{H}^\mu)$ is equal to $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$. In particular, for any $F \in W^{1,2}(\Gamma_{\mathbb{R}^d};\mu) \cap L^\infty(\Gamma_{\mathbb{R}^d};\mu)$ and $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, we have $F \in \mathbb{H}^\mu$, $D^\mu F = d^\mu F$ μ-a.e., and

(13) $D_\Lambda^\mu E_\mu[F|\mathcal{F}(\Lambda)] = E_\mu[d^\mu F|\mathcal{F}(\Lambda)] + E_\mu[(F - E_\mu[F|\mathcal{F}(\Lambda)])b|\mathcal{F}(\Lambda)], \quad \mu - a.e.$

Proof. Fix a $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$. We will first prove that $\mathcal{H}^\mu \subset W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ by showing that $\mathbb{H}^\mu \subset W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ and $D^\mu = d^\mu$ on $\mathbb{H}^\mu$. It is sufficient to show that for any $F \in \mathbb{H}^\mu$, $G \in \mathcal{F}C_b^\infty$, and $v = \sum_{j=1}^d v_j e_j \in V_0(\mathbb{R}^d)$

$$\int_{\Gamma_{\mathbb{R}^d}} \langle D^\mu F, Gv\rangle_{T\Gamma_{\mathbb{R}^d}} d\mu = -\int_{\Gamma_{\mathbb{R}^d}} F \mathrm{div}_\phi^\Gamma(Gv) d\mu.$$

We choose two arbitrary sequences $\{\Delta_l \in \mathcal{O}_c(\mathbb{R}^d)\}$ and $\{F_l \in H(\Delta_l;\mu)\}$ satisfying $F_l \to F$ in $L^2(\Gamma_{\mathbb{R}^d};\mu)$ and $D_{\Delta_l}^\mu F_l \xrightarrow{w} D^\mu F$ as $l \to \infty$. By Proposition 3.1 and Lemma 3.2, we get

$$\begin{aligned}
&\int_{\Gamma_{\mathbb{R}^d}} \langle D^\mu F, Gv\rangle_{T\Gamma_{\mathbb{R}^d}} d\mu \\
&= \lim_{l\to\infty} \int_{\Gamma_{\mathbb{R}^d}} \langle D_{\Delta_l}^\mu F_l, Gv\rangle_{T\Gamma_{\mathbb{R}^d}} d\mu \\
&= \lim_{l\to\infty} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Delta_l^n} \sum_{i=1}^n \left\langle \nabla_{x_i} F_{l\Delta_l}^n(x_1,\ldots,x_n), G\left(\sum_{i=1}^n \varepsilon_{x_i}\right) v(x_i)\right\rangle_{\mathbb{R}^d} \\
&\quad \cdot \sigma_{\Delta_l}^n(x_1,\ldots,x_n) m(dx_1)\cdots m(dx_n) \\
&= -\lim_{l\to\infty} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Delta_l^n} F_{l\Delta_l}^n(x_1,\ldots,x_n) \sum_{i=1}^n \left[\left\langle \nabla_{x_i} G\left(\sum_{i=1}^n \varepsilon_{x_i}\right), v(x_i)\right\rangle_{\mathbb{R}^d}\right. \\
&\quad \cdot \sigma_{\Delta_l}^n(x_1,\ldots,x_n) + G\left(\sum_{i=1}^n \varepsilon_{x_i}\right) \sum_{j=1}^d \partial_j v_j(x_i) \sigma_{\Delta_l}^n(x_1,\ldots,x_n) \\
&\quad \left. + G\left(\sum_{i=1}^n \varepsilon_{x_i}\right) \langle \nabla_{x_i} \sigma_{\Delta_l}^n(x_1,\ldots,x_n), v(x_i)\rangle_{\mathbb{R}^d}\right] m(dx_1)\cdots m(dx_n) \\
&= -\lim_{l\to\infty} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Delta_l^n} F_{l\Delta_l}^n(x_1,\ldots,x_n) \left\{\sum_{i=1}^n \left[\left\langle \nabla_{x_i} G\left(\sum_{i=1}^n \varepsilon_{x_i}\right), v(x_i)\right\rangle_{\mathbb{R}^d}\right.\right.
\end{aligned}$$

$$
+G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\sum_{j=1}^{d}\partial_j v_j(x_i)\Bigg] + (GL^{\phi}_{v\,\Delta_l})\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\Bigg\}
$$

$$
\cdot\sigma^n_{\Delta_l}(x_1,\ldots,x_n)m(dx_1)\cdots m(dx_n)
$$

$$
= -\int_{\Gamma_{\mathbb{R}^d}} F\mathrm{div}^{\Gamma}_{\phi}(Gv)d\mu.
$$

We now prove that $W^{1,2}_{\infty}(\Gamma_{\mathbb{R}^d};\mu) \subset \mathcal{H}^{\mu}$. It is sufficient to show that for any bounded function F in $W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$, $F \in \mathbb{H}^{\mu}$. To simplify the formulas, we denote $E_\mu[F|\mathcal{F}(\Lambda)](\gamma) = F_\Lambda(\gamma_\Lambda) = \sum_{n=0}^{\infty} F^n_\Lambda((s^n_\Lambda)^{-1}\gamma_\Lambda)$, $E_\mu[d^\mu F|\mathcal{F}(\Lambda)](\gamma) = \mathbb{F}_\Lambda(\gamma_\Lambda)$ and $E_\mu[(F-F_\Lambda)b|\mathcal{F}(\Lambda)](\gamma) = b^F_\Lambda(\gamma_\Lambda)$.

For an arbitrary $n_0 \in \mathbb{N}$, we let $\psi : \mathbb{R} \to [0,1]$ be an infinitely differentiable function such that $\psi(n_0) = 1$, $\psi'(n_0) = 0$ and $\psi(x) = 0$ for $|x - n_0| \geq \frac{1}{2}$. We let $\{\zeta_m\}_{1\leq m<\infty}$ be a (smooth) partition of unity for Λ and define $\psi_M(\gamma) := \psi(\langle\sum_{m=1}^{M}\zeta_m,\gamma\rangle)$ for $M \in \mathbb{N}$. Then, by [11, Proposition 4.5], $F\psi_M \in W^{1,2}_{\infty}(\Gamma_{\mathbb{R}^d};\mu)$. For any $G \in \mathcal{F}C^{\infty}_b(\Lambda)$ and $v = \sum_{j=1}^{d} v_j e_j \in V_0(\Lambda)$, we get from Lemma 3.2 that

$$
-\frac{1}{n_0!}\int_{\Lambda^{n_0}}(F^{n_0}_\Lambda\sigma^{n_0}_\Lambda)(x_1,\ldots,x_{n_0})\sum_{i=1}^{n_0}\left[\left\langle\nabla_{x_i}G\left(\sum_{i=1}^{n_0}\varepsilon_{x_i}\right), v(x_i)\right\rangle_{\mathbb{R}^d}\right.
$$

$$
\left.+G\left(\sum_{i=1}^{n_0}\varepsilon_{x_i}\right)\sum_{j=1}^{d}\partial_j v_j(x_i)\right] m(dx_1)\cdots m(dx_{n_0})
$$

$$
= -\lim_{M\to\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}(F^n_\Lambda\sigma^n_\Lambda)(x_1,\ldots,x_n)\psi_M\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)
$$

$$
\cdot\sum_{i=1}^{n}\left[\left\langle\nabla_{x_i}G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right), v(x_i)\right\rangle_{\mathbb{R}^d} + G\left(\sum_{i=1}^{n}\varepsilon_{x_i}\right)\sum_{j=1}^{d}\partial_j v_j(x_i)\right]
$$

$$
m(dx_1)\cdots m(dx_n)
$$

$$
= -\lim_{M\to\infty}\int_{\Gamma_{\mathbb{R}^d}} F_\Lambda\psi_M(\mathrm{div}^{\Gamma}_{\phi}(Gv) - L^{\phi}_v G)d\mu
$$

$$
= -\lim_{M\to\infty}\int_{\Gamma_{\mathbb{R}^d}} F\psi_M(\mathrm{div}^{\Gamma}_{\phi}(Gv) - L^{\phi}_v G)d\mu
$$

$$
= \lim_{M\to\infty}\left\{\int_{\Gamma_{\mathbb{R}^d}}\langle d^\mu(F\psi_M), Gv\rangle_{T\Gamma_{\mathbb{R}^d}}d\mu + \int_{\Gamma_{\mathbb{R}^d}}\psi_M FG(L^{\phi}_v - E_\mu[L^{\phi}_v|\mathcal{F}(\Lambda)])d\mu\right.
$$

$$
\left.+\int_{\Gamma_{\mathbb{R}^d}}\psi_M FG\frac{\langle D^\mu_\Lambda\varphi^2_\Lambda, v\rangle_{T\Gamma_{\mathbb{R}^d}}}{\varphi^2_\Lambda}d\mu\right\}
$$

$$
= \lim_{M\to\infty}\left\{\int_{\Gamma_{\mathbb{R}^d}}\langle\psi_M d^\mu F + F d^\mu\psi_M, Gv\rangle_{T\Gamma_{\mathbb{R}^d}}d\mu\right.
$$

$$+\int_{\Gamma_{\mathbb{R}^d}} \psi_M FG(L_v^\phi - E_\mu[L_v^\phi|\mathcal{F}(\Lambda)])d\mu + \int_{\Gamma_{\mathbb{R}^d}} \psi_M FG\frac{\langle D_\Lambda^\mu \varphi_\Lambda^2, v\rangle_{T\Gamma_{\mathbb{R}^d}}}{\varphi_\Lambda^2}d\mu\Bigg\}$$

$$= \int_{\Gamma_{\mathbb{R}^d}} \langle \psi(\langle 1,\cdot\rangle)\langle d^\mu F, Gv\rangle_{T\Gamma_{\mathbb{R}^d}}d\mu + \int_{\Gamma_{\mathbb{R}^d}} \psi(\langle 1,\cdot\rangle)FG(L_v^\phi - E_\mu[L_v^\phi|\mathcal{F}(\Lambda)])d\mu$$

$$+\int_{\Gamma_{\mathbb{R}^d}} \psi(\langle 1,\cdot\rangle)FG\frac{\langle D_\Lambda^\mu \varphi_\Lambda^2, v\rangle_{T\Gamma_{\mathbb{R}^d}}}{\varphi_\Lambda^2}d\mu$$

$$= \int_{\Gamma_{\mathbb{R}^d}} \psi(\langle 1,\cdot\rangle)\langle E_\mu[d^\mu F|\mathcal{F}(\Lambda)], Gv\rangle_{T\Gamma_{\mathbb{R}^d}}d\mu$$

$$+\int_{\Gamma_{\mathbb{R}^d}} \psi(\langle 1,\cdot\rangle)\langle E_\mu[(F-F_\Lambda)b|\mathcal{F}(\Lambda)], Gv\rangle_{T\Gamma_{\mathbb{R}^d}}d\mu$$

$$+\int_{\Gamma_{\mathbb{R}^d}} \psi(\langle 1,\cdot\rangle)FG\frac{\langle D_\Lambda^\mu \varphi_\Lambda^2, v\rangle_{T\Gamma_{\mathbb{R}^d}}}{\varphi_\Lambda^2}d\mu$$

$$= \frac{1}{n_0!}\int_{\Lambda^{n_0}} \Bigg\{\langle \mathbb{F}_\Lambda + b_\Lambda^F, Gv\rangle_{T_{\{x_1,\dots,x_{n_0}\}}\Gamma_{\mathbb{R}^d}}\sigma_\Lambda^{n_0}(x_1,\dots,x_{n_0}) + \sum_{i=1}^{n_0}\langle F_\Lambda^{n_0}(x_1,\dots,x_{n_0})$$

$$\cdot\nabla_{x_i}\sigma_\Lambda^{n_0}(x_1,\dots,x_{n_0}), G\left(\sum_{i=1}^{n_0}\varepsilon_{x_i}\right)v(x_i)\Bigg\rangle_{\mathbb{R}^d}\Bigg\} m(dx_1)\cdots m(dx_{n_0}).$$

Since both $G \in \mathcal{F}C_b^\infty(\Lambda)$ and $v \in V_0(\Lambda)$ are arbitrary, we conclude form the theorem of partition of unity that for $m^{\otimes n_0}$-a.e. $(x_1,\dots,x_{n_0})$,

$$(\nabla_{x_i}(F_\Lambda^{n_0}\sigma_\Lambda^{n_0})(x_1,\dots,x_{n_0}))_{1\le i\le n_0}$$
$$= (\mathbb{F}_\Lambda + b_\Lambda^F)\left(\sum_{i=1}^{n_0}\varepsilon_{x_i}\right)\sigma_\Lambda^{n_0}(x_1,\dots,x_{n_0})$$
$$+F_\Lambda^{n_0}(x_1,\dots,x_{n_0})(\nabla_{x_i}\sigma_\Lambda^{n_0}(x_1,\dots,x_{n_0}))_{1\le i\le n_0}.$$

Thus, for $\sigma_\Lambda^{n_0}m^{\otimes n_0}$-a.e. $(x_1,\dots,x_{n_0})$,

$$(\nabla_{x_i}F_\Lambda^{n_0}(x_1,\dots,x_{n_0}))_{1\le i\le n_0} = (\mathbb{F}_\Lambda + b_\Lambda^F)\left(\sum_{i=1}^{n_0}\varepsilon_{x_i}\right).$$

Since n_0 is arbitrary, $E_\mu[F|\mathcal{F}(\Lambda)] \in H(\Lambda;\mu)$ and (13) holds. Moreover, let $\{\triangle_l : \Lambda \subset \triangle_l \in \mathcal{B}_c(\mathbb{R}^d)\}$ be a sequence tending to $\mathbb{R}^d$. Then, for any $V \in \mathcal{V}\mathcal{F}C_b^\infty$, we have

$$\begin{aligned}\int_{\Gamma_{\mathbb{R}^d}} |\langle D_{\triangle_l}^\mu E_\mu[F|\mathcal{F}(\triangle_l)] - d^\mu F, V\rangle_{T\Gamma_{\mathbb{R}^d}}|d\mu &= \int_{\Gamma_{\mathbb{R}^d}} |\langle b_{\triangle_l}^F, V\rangle_{T\Gamma_{\mathbb{R}^d}}|d\mu \\ &\le ||(F-F_{\triangle_l})b||_-||V||_+ \\ &\to 0, \quad \text{as } l\to\infty.\end{aligned}$$

Thus, $F \subset \mathcal{H}^\mu$ and $D^\mu F = d^\mu F$ μ-a.e. The proof is therefore done.

Theorem 3.4. Suppose that a superstable, lower regular and integrable pair potential ϕ is nonnegative and satisfies conditions (D) and (P). Let $\triangle_\phi^\Gamma$ be the operator on $L^2(\Gamma_{\mathbb{R}^d};\mu)$ with domain $\mathcal{F}C_b^\infty$ defined by $\triangle_\phi^\Gamma := \mathrm{div}_\phi^\Gamma \nabla^\Gamma$. Then, $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is a Dirichlet form on $L^2(\Gamma_{\mathbb{R}^d};\mu)$ and its generator is the maximum Markovian self-adjoint extension of $(\triangle_\phi^\Gamma, \mathcal{F}C_b^\infty)$.

Proof. By [4, Theorem 3.1], it is sufficient to prove that $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is a Dirichlet form on $L^2(\Gamma_{\mathbb{R}^d};\mu)$, i.e., $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ satisfies the contraction property. According to [5, Theorem 3.1.1], one needs to prove that for every function $T:\mathbb{R}\to\mathbb{R}$ with $T(0)=0$ and $|T(s)-T(t)|\le|s-t|$ for all $s,t\in\mathbb{R}$, one has $T(F)\in\mathbb{H}^\mu$ and

$$\mathcal{E}^\mu(T(F),T(F)) \le \mathcal{E}^\mu(F,F), \quad \forall F\in\mathbb{H}^\mu. \tag{14}$$

In fact, if $F\in\mathbb{H}^\mu$ then there exist an increasing sequence $\{\triangle_l\in\mathcal{O}_c(\mathbb{R}^d)\}$ and a sequence of functions $\{F_l\in H(\triangle_l;\mu)\}$ satisfying $F_l\to F$ in $L^2(\Gamma_{\mathbb{R}^d};\mu)$ and $D^\mu_{\triangle_l}F_l \xrightarrow{w} D^\mu F$ as $l\to\infty$. Obviously, $T(F_l)\in H(\triangle_l;\mu)$, $T(F_l)\to T(F)$ in $L^2(\Gamma_{\mathbb{R}^d};\mu)$ and $D^\mu_{\triangle_l}T(F_l) = T'(F_l)D^\mu_{\triangle_l}(F_l) \xrightarrow{w} T'(F)D^\mu F$ as $l\to\infty$. Therefore, $F\in\mathbb{H}^\mu$ by the definition of $\mathbb{H}^\mu$ and (14) holds. This completes the proof.

We end this section by giving an application of the above results. It is shown in [3] that if the pair potential ϕ is three times differentiable and decreases subexponentially, then $\triangle_\phi^\Gamma$ is essentially self-adjoint on $D_0^2(\overline{\Omega})$, a class of bounded smooth local functions on $\Gamma_{\mathbb{R}^d}$ which contains $\mathcal{F}C_b^\infty$. For the definition of the function space $D_0^2(\overline{\Omega})$ and more details we refer the reader to [3].

Theorem 3.5. [3, Theorem II.17] Suppose that a pair potential ϕ is superstable and lower regular. Furthermore, assume that $\phi\in C_b^3(\mathbb{R}^d)$ and there exist a constant $c_0>0$ and a function α that satisfies the following conditions:
(i) $\alpha(0)\ge 1$ and $\alpha(\lambda)\to\infty$ as $\lambda\to\infty$.
(ii) $\alpha'(\lambda)\le\alpha(\lambda)/(1+\lambda)$ for some $\lambda\ge 0$, and $\alpha''(\lambda)\ge -c_0/(1+\lambda)$.
(iii) For all $x\in\mathbb{R}^d$, $i,j,k=1,\ldots,d$,

$$\left|\frac{\partial\phi}{\partial x^i}(x)\right| + \left|\frac{\partial^2\phi}{\partial x^j\partial x^i}(x)\right| + \left|\frac{\partial^3\phi}{\partial x^k\partial x^j\partial x^i}(x)\right| \le \exp\{-c_0\ln(1+|x|^2)\alpha(1+|x|^2)\}.$$

Then the Dirichlet operator $\triangle_\phi^\Gamma$ is essentially self-adjoint on the domain $D_0^2(\overline{\Omega})$.

In [12], two useful "intermediate" spaces $H_0^{1,2}(\Gamma_{\mathbb{R}^d};\mu)\subset\mathcal{F}^{(c)}\subset\mathcal{F}\subset W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ were introduced. If a pair potential ϕ is nonnegative and satisfies all the conditions in Theorem 3.5, then by virtue of Proposition 3.1, integration by parts, and Theorem 3.3, one can show that the generators of $(\mathcal{E}^\mu,\mathcal{F}^{(c)})$, $(\mathcal{E}^\mu,\mathcal{F})$ and $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ are all self-adjoint extensions of $(\triangle_\phi^\Gamma, D_0^2(\overline{\Omega}))$. Therefore, by Theorem 3.5,

$$\mathcal{F}^{(c)} = \mathcal{F} = W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu). \tag{15}$$

Some nontrivial results can be obtained from (15). Firstly, we point that the maximum Dirichlet form $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is a quasi-regular Dirichlet form on $L^2(\Gamma_{\mathbb{R}^d};\mu)$ since $(\mathcal{E}^\mu, \mathcal{F}^{(c)})$ is quasi-regular by [12, Corollary 3.4]. Secondly, any function in $W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ can be approximated by smooth local functions with respect to the Dirichlet form norm. Note that this is a priori not known for more general situation. Thirdly, let $((X_t)_{t\geq 0}, (P_\gamma)_{\gamma\in\overline{\Gamma}_{\mathbb{R}^d}})$ be the conservative diffusion process associated with $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ and denote the generator of $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ by L^μ. Then we have the following analogue of [11, Proposition 9.2], which is an improvement of [11, Theorem 9.13].

Proposition 3.6. The following assertions are equivalent:
(i) $P_\mu := \int_{\Gamma_{\mathbb{R}^d}} P_\gamma \mu(d\gamma)$ is (time) ergodic.
(ii) $(e^{-tL^\mu})_{t>0}$ is irreducible.
(iii) If $F \in L^2(\Gamma_{\mathbb{R}^d};\mu)$ such that $e^{-tL^\mu}F = F$ for all $t>0$, then $F = const.$
(iv) $(e^{-tL^\mu})_{t>0}$ is ergodic.
(v) If $F \in D(L^\mu)$ with $L^\mu F = 0$, then $F = const.$
(vi) $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$ is irreducible.
(vii) μ is an extreme point of the set of all Ruelle measures with the pair potential ϕ.

4 Markov Processes associated with $(\mathcal{E}^\mu, W_\infty^{1,2}(\Gamma_{\mathbb{R}^d};\mu))$

Throughout this section, we suppose that the pair potential ϕ satisfies the conditions in Theorem 3.4. Let $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$. We define

$$\begin{cases} D(\mathcal{E}_\Lambda^\mu) := \{F \in L^2(\Gamma_\Lambda;\mu_\Lambda) : F\circ p_\Lambda \in H(\Lambda;\mu)\}, \\ \mathcal{E}_\Lambda^\mu(F,G) := \displaystyle\int_{\Gamma_{\mathbb{R}^d}} \langle D_\Lambda^\mu(F\circ p_\Lambda), D_\Lambda^\mu(G\circ p_\Lambda)\rangle_{T\Gamma_{\mathbb{R}^d}} d\mu,\ \ F,G\in D(\mathcal{E}_\Lambda^\mu) \end{cases}$$

and

$$\begin{cases} D(\mathcal{E}_\Lambda^{\mu,0}) := \overline{\mathcal{F}C_b^\infty|_{\Gamma_\Lambda}}, \\ \mathcal{E}_\Lambda^{\mu,0}(F,G) := \mathcal{E}_\Lambda^\mu(F,G),\ \ F,G\in D(\mathcal{E}_\Lambda^{\mu,0}), \end{cases}$$

where $\mathcal{F}C_b^\infty|_{\Gamma_\Lambda}$ denotes the restriction of $\mathcal{F}C_b^\infty$ on Γ_Λ and $\overline{\mathcal{F}C_b^\infty|_{\Gamma_\Lambda}}$ the closure of $\mathcal{F}C_b^\infty|_{\Gamma_\Lambda}$ in $D(\mathcal{E}_\Lambda^\mu)$ with respect to the Dirichlet form norm.

Lemma 4.1. Suppose that a pair potential ϕ satisfies the conditions in Theorem 3.4. Then, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$,

$$(\mathcal{E}_\Lambda^\mu, D(\mathcal{E}_\Lambda^\mu)) = (\mathcal{E}_\Lambda^{\mu,0}, D(\mathcal{E}_\Lambda^{\mu,0})).$$

Proof. Let $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$. We first set $\mu = \pi_{zm}$ and consider the free case. For $n \in \mathbb{N}$, we let ψ_n be an infinitely differentiable function with support in $(n-1, n+1)$ such that $\psi_n(n) = 1$. Note that $\Psi_n := \psi_n(\langle 1, \cdot \rangle) \in \mathcal{F}C_b^\infty|_{\Gamma_\Lambda}$ and

$$\Psi_n(\gamma) = \begin{cases} 1, & \gamma \in \Gamma_\Lambda^{(n)}, \\ 0, & \gamma \in \Gamma_\Lambda \backslash \Gamma_\Lambda^{(n)}. \end{cases}$$

Let $F \in D(\mathcal{E}_\Lambda^{\pi_{zm}})$. We denote by $\overline{\Lambda}$ the closure of Λ and $C^\infty(\overline{\Lambda})$ the infinitely differentiable functions on $\overline{\Lambda}$. Since $C^\infty(\overline{\Lambda})$ is dense in the (1,2)-Sobolev space $H^{1,2}(\Lambda)$, one finds that $\Psi_n F \in \Psi_n \overline{\mathcal{F}C_b^\infty|_{\Gamma_\Lambda}} := \{\Psi_n G : G \in \overline{\mathcal{F}C_b^\infty|_{\Gamma_\Lambda}}\}$ for each $n \in \mathbb{N}$. Then, $F \in D(\mathcal{E}_\Lambda^{\pi_{zm},0})$. Since $F \in D(\mathcal{E}_\Lambda^{\pi_{zm}})$ is arbitrary, $(\mathcal{E}_\Lambda^{\pi_{zm}}, D(\mathcal{E}_\Lambda^{\pi_{zm}})) = (\mathcal{E}_\Lambda^{\pi_{zm},0}, D(\mathcal{E}_\Lambda^{\pi_{zm},0}))$.

We now let μ be a general Ruelle measure with the pair potential ϕ. Note that by Proposition 3.1, $\varphi_\Lambda \in D(\mathcal{E}_\Lambda^{\pi_{zm}})$. Then, $\varphi_\Lambda \in D(\mathcal{E}_\Lambda^{\pi_{zm},0})$. For $l \in \mathbb{N}$ we define

$$\varphi_l := a_l(\ln \varphi_\Lambda),$$

where we fix $a_l \in C_0^\infty(\mathbb{R})$ such that $1_{[-l,l]} \leq a_l \leq 1_{[-l-1,l+1]}$ and $|a_l'| \leq 2$. Then $\varphi_l \in D(\mathcal{E}_\Lambda^{\pi_{zm},0})$. By (2), it is easy to see that $\Psi_n \varphi_l \in D(\mathcal{E}_\Lambda^{\mu,0})$ for each $n \in \mathbb{N}$. Thus $\varphi_l \in D(\mathcal{E}_\Lambda^{\mu,0})$. Following the same argument as in [14, Proof of Theorem 2.3], one can show that the following claims hold for any bounded function $u \in D(\mathcal{E}_\Lambda^\mu)$:
Claim 1. $\varphi_l u \to u$ in $D(\mathcal{E}_\Lambda^\mu)$ as $l \to \infty$.
Claim 2. Let $l \in \mathbb{N}$, then there exists $\{u_n \in \mathcal{F}C_b^\infty|_{\Gamma_\Lambda}\}$ such that $\varphi_{l+1} u_n \to \varphi_l u$ in $D(\mathcal{E}_\Lambda^\mu)$ as $n \to \infty$.
Claim 3. Let $v \in \mathcal{F}C_b^\infty|_{\Gamma_\Lambda}$, then $\varphi_l v \in D(\mathcal{E}_\Lambda^{\mu,0})$ for each $l \in \mathbb{N}$.
Therefore $u \in D(\mathcal{E}_\Lambda^{\mu,0})$. This completes the proof.

Let $f : \mathbb{R} \to [0,1]$ be an infinitely differentiable function such that

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

For $n \in \mathbb{N}$, we define a cut-off function by $f_n(x) = f(|x|/n)$ for $x \in \mathbb{R}^d$. Moreover, for $F, G \in \mathcal{F}C_b^\infty$ we define

$$\mathcal{E}^{\mu,n}(F,G) := \int_{\Gamma_{\mathbb{R}^d}} \sum_{x \in \gamma} \langle \nabla_x F(\gamma), \nabla_x G(\gamma) \rangle_{\mathbb{R}^d} f_n^2(x) \mu(d\gamma).$$

Similar to [2, Proposition 5.1], one sees that $(\mathcal{E}^{\mu,n}, \mathcal{F}C_b^\infty)$ is closable and its closure $(\mathcal{E}^{\mu,n}, D(\mathcal{E}^{\mu,n}))$ is a symmetric Dirichlet form on $L^2(\Gamma_{\mathbb{R}^d}; \mu)$. We denote the (1,1)-capacities associated with $\{(\mathcal{E}^{\mu,n}, D(\mathcal{E}^{\mu,n}))\}$, $(\mathcal{E}^\mu, H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu))$ by $\{\mathrm{Cap}_{1,1}^n\}$, $\mathrm{Cap}_{1,1}$, respectively. Since for arbitrary $F \in H_0^{1,2}(\Gamma_{\mathbb{R}^d}; \mu)$, $\mathcal{E}^{\mu,n}(F,F) \leq \mathcal{E}^\mu(F,F)$, $\mathrm{Cap}_{1,1}^n(U) \leq \mathrm{Cap}_{1,1}(U)$ for any open subset of $\Gamma_{\mathbb{R}^d}$. Therefore, $(\mathcal{E}^{\mu,n}, D(\mathcal{E}^{\mu,n}))$ is a quasi-regular and local Dirichlet form.

We recall below the definition of Mosco convergence of bilinear forms and refer the reader to [9] for more details.

Definition 4.2. Let E be a Hausdorff topological space and λ a σ-finite measure on its Borel σ-algebra $\mathcal{B}(E)$. A sequence of symmetric bilinear forms $\{(\mathcal{A}^n, D(\mathcal{A}^n))\}$ on $L^2(E;\lambda)$ is said to converge to a form $(\mathcal{A}, D(\mathcal{A}))$ in the sense of Mosco convergence if and only if the following conditions are satisfied:
(i) If $\{u_n \in D(\mathcal{A}^n)\}$ and $u \in L^2(E;\lambda)$ such that $u_n \xrightarrow{w} u$ in $L^2(E;\lambda)$ as $n \to \infty$ and $\liminf_{n\to\infty} \mathcal{A}^n(u_n, u_n) < \infty$, then $u \in D(\mathcal{A})$ and $\mathcal{E}(u,u) \leq \liminf_{n\to\infty} \mathcal{A}^n(u_n,u_n)$.
(ii) For any $u \in D(\mathcal{A})$, there exists $\{u_n \in D(\mathcal{A}^n)\}$ such that $u_n \to u$ in $L^2(E;\lambda)$ as $n \to \infty$ and $\mathcal{A}(u,u) \geq \limsup_{n\to\infty} \mathcal{A}^n(u_n,u_n)$.

Mosco proved in [9] that Mosco convergence of a sequence of densely defined symmetric bilinear forms is equivalent to the convergence, in the strong operator sense, of the sequence of semigroups associated with the closures of the corresponding forms.

Theorem 4.3. Suppose that a pair potential ϕ satisfies the conditions in Theorem 3.4. Then, as $n \to \infty$, the sequence of Dirichlet forms $(\mathcal{E}^{\mu,n}, D(\mathcal{E}^{\mu,n}))$ converges to $(\mathcal{E}^\mu, W^{1,2}_\infty(\Gamma_{\mathbb{R}^d};\mu))$ in the sense of Mosco convergence.

Proof. First note that it is sufficient to prove that as $n \to \infty$, $\{(\mathcal{E}^{\mu,n}, \mathcal{F}C_b^\infty)\}$ converges to $(\mathcal{E}^\mu, W^{1,2}_\infty(\Gamma_{\mathbb{R}^d};\mu))$ in the sense of Mosco convergence. We suppose that $\{F_n \in \mathcal{F}C_b^\infty\}$ and $F \in L^2(\Gamma_{\mathbb{R}^d};\mu)$ satisfy $F_n \xrightarrow{w} F$ in $L^2(\Gamma_{\mathbb{R}^d};\mu)$ as $n \to \infty$, and $\liminf_{n\to\infty} \mathcal{E}^{\mu,n}(F_n,F_n) = M < \infty$. Let $V \in \mathcal{V}\mathcal{F}C_b^\infty$. Then

$$\begin{aligned}
\left|\int_{\Gamma_{\mathbb{R}^d}} F \mathrm{div}^\Gamma_\phi V d\mu\right| &= \lim_{n\to\infty}\left|\int_{\Gamma_{\mathbb{R}^d}} F_n \mathrm{div}^\Gamma_\phi V d\mu\right| \\
&= \lim_{n\to\infty}\left|\int_{\Gamma_{\mathbb{R}^d}} \sum_{x\in\gamma}\langle \nabla_x F_n(\gamma), V(\gamma)\rangle_{\mathbb{R}^d} f_n^2(x)\mu(d\gamma)\right| \\
&\leq M^{1/2}||V||_{L^2(\Gamma_{\mathbb{R}^d}\to T\Gamma_{\mathbb{R}^d};\mu)}.
\end{aligned}$$

Thus, $F \in W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ and $\mathcal{E}^\mu(F,F) \leq \liminf_{n\to\infty} \mathcal{E}^{\mu,n}(F_n,F_n)$. For $m \in \mathbb{N}$, let $\psi_m : \mathbb{R} \to [-(m+1), m+1]$ be an infinitely differentiable function such that $|\psi_m'| \leq 1$ and

$$\psi_m(x) = \begin{cases} x, & \text{if } |x| \leq m, \\ (m+1)\mathrm{sgn} x, & \text{if } |x| \geq m+2. \end{cases}$$

Using a similar argument, one finds that $\psi_m(F) \in W^{1,2}(\Gamma_{\mathbb{R}^d};\mu)$ and $\mathcal{E}^\mu(\psi_m(F), \psi_m(F)) \leq \liminf_{n\to\infty} \mathcal{E}^{\mu,n}(F_n,F_n)$. Therefore, $F \in W^{1,2}_\infty(\Gamma_{\mathbb{R}^d};\mu)$ by [7, Lemma 2.12].

We now prove that condition (ii) holds. Without loss of generality, we assume that $F \in W^{1,2}(\Gamma_{\mathbb{R}^d};\mu) \cap L^\infty(\Gamma_{\mathbb{R}^d};\mu)$. For $n \in \mathbb{N}$, let B_n denote the open ball in $\mathbb{R}^d$ with radius n, centered at the origin. By Theorem 3.3, $E_\mu[F|\mathcal{F}(B_n)] \to F$ in $L^2(\Gamma_{\mathbb{R}^d};\mu)$ and $||D^\mu_{B_n}E_\mu[F|\mathcal{F}(B_n)] - d^\mu F||_- \to 0$ as $n \to \infty$. Then, by Lemma 4.1, one can select a sequence of functions $\{F_n \in \mathcal{F}C_b^\infty\}$ such that $F_n \circ p_{B_{2n}} \to F$ in $L^2(E;\mu)$ and $||D^\mu_{B_{2n}}(F_n \circ p_{B_{2n}}) - d^\mu F||_- \to 0$ as $n \to \infty$. For an $n \in \mathbb{N}$, set $F_n = g(\langle f_1, \cdot\rangle, \dots \langle f_m, \cdot\rangle)$ for some $g \in C_b^\infty(\mathbb{R}^m)$ and $f_1, \dots, f_m \in C_0^\infty(\mathbb{R}^d)$. Let $\{\phi_k\}$ be a decreasing sequence of functions in $C_0^\infty(\mathbb{R}^d)$ such that $\phi_k|_{B_{2n}} = 1$ and $\phi_k \to \chi_{B_{2n}}$ (the characteristic function of B_{2n}) as $k \to \infty$. We define $G_n := g(\langle f_1\phi_k, \cdot\rangle, \dots \langle f_m\phi_k, \cdot\rangle)$ for some large enough k, so $\{G_n\}$ is a sequence of functions in $\mathcal{F}C_b^\infty$ such that $G_n \to F$ in $L^2(E;\mu)$ and $\mathcal{E}^\mu(F,F) \geq \limsup_{n\to\infty} \mathcal{E}^{\mu,n}(G_n, G_n)$ as $n \to \infty$. This completes the proof.

We define $\Omega := C([0,\infty) \to \overline{\Gamma}_{\mathbb{R}^d})$, $X_t(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \Omega$. For $n \in \mathbb{N}$, we let $M^n = (X_t, (P^n_\gamma)_{\gamma\in\overline{\Gamma}_{\mathbb{R}^d}})$ be the conservative diffusion process associated with the Dirichlet form $(\mathcal{E}^{\mu,n}, D(\mathcal{E}^{\mu,n}))$.

Theorem 4.4. Suppose that a pair potential ϕ satisfies the conditions in Theorem 3.4. Then, for μ-a.e. $\gamma \in \overline{\Gamma}_{\mathbb{R}^d}$, the sequence $\{P^n_\gamma\}$ converges weakly to a probability measure P_γ on Ω. The $\overline{\Gamma}_{\mathbb{R}^d}$-valued process $(X_t, (P_\gamma)_{\gamma\in\overline{\Gamma}_{\mathbb{R}^d}})$ is associated with the maximum Dirichlet form $(\mathcal{E}^\mu, W^{1,2}_\infty(\Gamma_{\mathbb{R}^d};\mu))$.

Proof. By Theorem 4.3, $\{(\mathcal{E}^{\mu,n}, D(\mathcal{E}^{\mu,n}))\}$ converges to $(\mathcal{E}^\mu, W^{1,2}_\infty(\Gamma_{\mathbb{R}^d};\mu))$ in the sense of Mosco convergence. As a consequence, the sequence of associated semigroups on $L^2(\overline{\Gamma}_{\mathbb{R}^d};\mu)$ converges in the strong operator sense. By the Markov property, one can check that for μ-a.e. $\gamma \in \overline{\Gamma}_{\mathbb{R}^d}$ the finite dimensional distributions of $\{P^n_\gamma\}$ converge. It is therefore sufficient to show that for μ-a.e. $\gamma \in \overline{\Gamma}_{\mathbb{R}^d}$, $\{P^n_\gamma\}$ is tight on Ω in Prohorov topology.

Let $\{F_i\}$ be a countable subset of $\mathcal{F}C_b^\infty$ which is dense in $\mathcal{F}C_b^\infty$ with respect to the uniform norm. We denote

$$c_i := \int_{\Gamma_{\mathbb{R}^d}} \langle \nabla^\Gamma F_i, \nabla^\Gamma F_i\rangle^2_{T\Gamma_{\mathbb{R}^d}} d\mu \quad \text{ for } 1 \leq i < \infty$$

and define a pseudo metric ρ on $\overline{\Gamma}_{\mathbb{R}^d}$ as follows:

$$\rho(\gamma_1,\gamma_2) = \sum_{i=1}^\infty \frac{|F_i(\gamma_1) - F_i(\gamma_2)|}{2^i(||F_i||_\infty + c_i + 1)}.$$

Note that the topology of $\overline{\Gamma}_{\mathbb{R}^d}$ induced by ρ is equivalent to the original topology on $\overline{\Gamma}_{\mathbb{R}^d}$. Following an observation of Kuwae and Uemura [6], we only need to prove that for μ-a.e. $\gamma \in \overline{\Gamma}_{\mathbb{R}^d}$, $\{P^n_\gamma\}$ is tight on Ω in the sense of Prohorov topology induced by ρ since the notion of tightness on Ω depends only upon the topology of $\overline{\Gamma}_{\mathbb{R}^d}$ not on a particular metric, whenever it induces the same original topology.

We denote by E^n_μ the expectation with respect to $P^n_\mu := \int_{\Gamma_{\mathbb{R}^d}} P^n_\gamma \mu(d\gamma)$. Then, for any $0 \leq s, t \leq T$, we have

$$E^n_\mu[\rho(X_s, X_t)^4] = E^n_\mu\left[\left(\sum_{i=1}^{\infty} \frac{|F_i(X_s) - F_i(X_t)|}{2^i(||F_i||_\infty + c_i + 1)}\right)^4\right]$$
$$\leq \sum_{i=1}^{\infty} \frac{E^n_\mu[|F_i(X_s) - F_i(X_t)|^4]}{2^i(||F_i||_\infty + c_i + 1)^4}.$$

Since the diffusion process M^n is conservative, we can apply the Lyons-Zheng decomposition to F_i for each i (cf. [5, Theorem 5.7.1]),

$$F_i(X_s) - F_i(X_t) = \frac{1}{2}(M_s^{[F_i],n} - M_t^{[F_i],n}) + \frac{1}{2}(M_{T-s}^{[F_i],n}(r_T) - M_{T-t}^{[F_i],n}(r_T)), \quad P^n_\mu - a.e.,$$

where $M^{[F_i],n}$ is a martingale additive functional of finite energy and r_T a time reversal operator at T defined by $X_t(r_T) = X_{T-t}, 0 \leq t \leq T$. By symmetry, for some constant C,

$$\begin{aligned}
E^n_\mu\ &[(|F_i(X_s) - F_i(X_t)|)^4] \\
&\leq 2E^n_\mu[(|F_i(M_s^{[F_i],n}) - F_i(M_t^{[F_i],n})|)^4] + 2E^n_\mu[(|F_i(M_{T-s}^{[F_i],n}) - F_i(M_{T-t}^{[F_i],n})|)^4] \\
&\leq CE^n_\mu\left[\left(\int_s^t \sum_{x\in X_u} \langle \nabla_x F_i(X_u), \nabla_x F_i(X_u)\rangle_{\mathbb{R}^d} f_n^2(x) du\right)^2\right] \\
&\quad + CE^n_\mu\left[\left(\int_{T-t}^{T-s} \sum_{x\in X_u} \langle \nabla_x F_i(X_u), \nabla_x F_i(X_u)\rangle_{\mathbb{R}^d} f_n^2(x) du\right)^2\right] \\
&\leq C(t-s)\int_s^t E^n_\mu[\langle \nabla F_i(X_u), \nabla F_i(X_u)\rangle^2]du \\
&\quad + C(t-s)\int_{T-t}^{T-s} E^n_\mu[\langle \nabla F_i(X_u), \nabla F_i(X_u)\rangle^2]du \\
&= 2C(t-s)^2 c_n.
\end{aligned}$$

Thus

$$\sup_n E^n_\mu[\rho(X_s, X_t)^4] \leq 2C(t-s)^2.$$

According to Kolmogorov's criterion,

$$\lim_{\delta\to 0} \sup_n P^n_\mu\left(\sup_{\substack{0\leq s,t\leq T\\ |t-s|\leq\delta}} \rho(X_s, X_t) > r\right) = 0 \quad \text{for all } r > 0.$$

On the other hand, let $\mu^n_{t_0}$ denote the distribution of X_{t_0} under P^n_μ. Note that by [7, Theorem 5.4], for $U \subset \Gamma_{\mathbb{R}^d}$, U open,

$$E^n_\mu[e^{-\sigma_U}] = \mathrm{Cap}^n_{1,1}(U),$$

where σ_U is the first hitting time of U. Since $(\mathcal{E}^\mu, H^{1,2}_0(\Gamma_{\mathbb{R}^d};\mu))$ is quasi-regular and $\mathrm{Cap}^n_{1,1}(U) \leq \mathrm{Cap}_{1,1}(U)$, the capacities $\{\mathrm{Cap}^n_{1,1}\}$ and then the measures $\{\mu^n_{t_0}\}$ are tight for each $t_0 \geq 0$. Thus $\{P^n_\mu\}$ is tight on Ω and therefore for μ-a.e. $\gamma \in \overline{\Gamma}_{\mathbb{R}^d}$, $\{P^n_\gamma\}$ is tight on Ω. This completes the proof.

References

[1] Albeverio, S., Kondratiev, Yu. G., Röckner, M.: Analysis and geometry on configuration spaces. J. Funct. Anal. **154** (1998), 444–500

[2] Albeverio, S., Kondratiev, Yu. G., Röckner, M.: Analysis and geometry on configuration spaces: The Gibbsian case. J. Funct. Anal. **157** (1998), 242–291

[3] Choi, V., Park, Y. M., Yoo, H. J.: Dirichlet forms and Dirichlet operators for infinite particle systems: essential self-adjointness. J. Math. Phys. **39** (1998), 6509-6536

[4] Eberle, A.: Uniqueness and Non-uniqueness of Semigroups Generated by Singular Diffusion Operators. Lect. Notes in Math. 1718. Springer, Berlin 1999

[5] Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet Forms and Symmetric Markov Processes. Walter de Gruyter, Berlin-New York 1994

[6] Kuwae, K., Uemura, T.: Weak convergence of symmetric diffusion processes. Prob. Th. Rel. Fields. **109** (1997), 159-182

[7] Ma, Z. M., Röckner, M.: Introduction to the Theory of (Non-symmetric) Dirichlet Forms. Springer, Berlin 1992

[8] Ma, Z. M., Röckner, M.: Construction of diffusions on configuration spaces. Osaka J. Math. **37**(2000), 273-314

[9] Mosco, U.: Composite media and asymptotic Dirichlet forms. J. Funct. Anal. **123** (1994), 368-421

[10] Osada, H.: Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions. Commun. Math. Phys. **176** (1996), 117–131

[11] Röckner, M.: Stochastic analysis on configuration spaces: basic ideas and recent results. In: New Directions in Dirichlet Forms, AMS/IP Stud. Adv. Math. 8, pp. 157–231. Amer. Math. Soc., Providence, RI, 1998

[12] Röckner, M., Schied, A.: Rademacher's theorem on configuration spaces and applications. J. Funct. Anal. **169** (1999), 325–356

[13] Röckner, M., Schmuland, B.: A support property for infinite-dimensional interacting diffusion processes. C. R. Acad. Sci. Paris. Sér. 1 Math. **326** (1998), 359–364

[14] Röckner, M., Zhang, T. S.: Uniqueness of generalized Schödinger operators and applications. J. Funct. Anal. **105** (1992), 187-231

[15] Ruelle, D.: Superstable interactions in classical statistical mechanics. Commun. Math. Phys. **18** (1970), 127–159

[16] Yoshida, M. W.: Construction of infinite dimensional interacting diffusion processes through Dirichlet forms. Prob. Th. Rel. Fields. **106** (1996), 265–297

Byron Schmuland and Wei Sun,
Department of Mathematical and Statistical Sciences,
University of Alberta,
Edmonton, Alberta, Canada T6G 2G1
`schmu@math.ualberta.ca` `wsun@math.ualberta.ca`