Positivity preserving forms have the Fatou property

Byron Schmuland
University of Alberta

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Abstract. We prove that if $(\mathcal{E}, D(\mathcal{E}))$ is a positivity preserving form on $L^2(E;m)$, and if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $D(\mathcal{E})$ converging m-almost everywhere to $u \in L^2(E;m)$, then $\mathcal{E}(u,u) \le \liminf_n \mathcal{E}(u_n,u_n)$.

Key words: Dirichlet form, positivity preserving, lower semi-continuity

If $(u_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(E;m)$ converging m-almost everywhere to u, then Fatou's lemma says that $(u,u)_{L^2} \le \liminf_n (u_n,u_n)_{L^2}$, where we set $(u,u)_{L^2} = \infty$ if $u \notin L^2(E;m)$. The corresponding result, where a Dirichlet form replaces the inner product, was used by Silverstein [5; Lemma 1.7] and by Fukushima, Oshima, and Takeda [2; Theorem 1.5.2] to define extended Dirichlet space and study time changes for symmetric Markov processes. However, their proofs require that E is a locally compact, separable, metric space, and that m is a positive Radon measure with full support. The purpose of this note is to drop the restrictions on E and m. In Proposition 1, we prove the Fatou property for symmetric Dirichlet forms, and in Proposition 2 we generalize the result to positivity preserving forms.

The Fatou property simply means that the function $v \mapsto \mathcal{E}(v,v)$ is lower semi-continuous on $L^2(E;m)$ with respect to the topology of convergence in m-measure. By way of comparison, the lower semi-continuity of $v \mapsto \mathcal{E}(v,v)$ with respect to L^2 convergence is equivalent (see [4]) to the form $\mathcal{E}$ being closed. Not every closed form has the Fatou property, but this stronger result holds when the form is positivity preserving.

DEFINITION 1. A densely defined, closed, non-negative definite, bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called a *positivity preserving form* if $v \in D(\mathcal{E})$ implies that $v^+ \in D(\mathcal{E})$ and that $\mathcal{E}(v^+, v^-) \le 0$. We set $\mathcal{E}(v,v) = \infty$ for $v \in L^2(E;m) \setminus D(\mathcal{E})$.

Note 1. If a positivity preserving form $(\mathcal{E}, D(\mathcal{E}))$ is symmetric, then it has a symmetric resolvent $(G_\lambda)_{\lambda>0}$. Each G_λ is a bounded, symmetric operator mapping $L^2(E;m)$ into $D(\mathcal{E})$ and such that $\mathcal{E}(G_\lambda v, w) + \lambda (G_\lambda v, w)_{L^2} = (v,w)_{L^2}$ for all $v \in L^2(E;m)$ and $w \in D(\mathcal{E})$. These operators are also positivity preserving in the sense that $v \ge 0$ implies $G_\lambda v \ge 0$. For $v \ge 0$ we define $\lim_{\lambda\to 0} \uparrow G_\lambda v := Gv \le \infty$. Finally, we note that $\mathcal{E}$ can be approximated by continuous positivity preserving forms. Setting $\mathcal{E}^\lambda(v,w) := \lambda(v - \lambda G_\lambda v, w)_{L^2}$, then $\mathcal{E}^\lambda(v,v) \uparrow \mathcal{E}(v,v)$ as $\lambda \to \infty$, for all $v \in L^2(E;m)$ [3; Chapter I].

DEFINITION 2. A densely defined, closed, non-negative definite, bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called a *Dirichlet form* if $v \in D(\mathcal{E})$ implies that $v \wedge 1 \in D(\mathcal{E})$ and that $\mathcal{E}(v \wedge 1, v \wedge 1) \le \mathcal{E}(v \wedge 1, v)$.

Note 2. Every Dirichlet form is positivity preserving, and the corresponding resolvent operators have the *Markov* property: if $0 \le v \le 1$, then $0 \le \lambda G_\lambda v \le 1$.

DEFINITION 3. A positivity preserving form $(\mathcal{E}, D(\mathcal{E}))$ is called *transient* if there exists a function $\rho \in L^2(E;m)$ such that ρ is strictly positive m-almost everywhere and $\int_E \rho\,|v|\,dm \le \mathcal{E}(v,v)^{1/2}$ for all $v \in D(\mathcal{E})$.

Note 3. If $(\mathcal{E}, D(\mathcal{E}))$ is transient, then for every $\lambda > 0$, we have

$$\begin{aligned}\int_E \rho\, G_\lambda \rho\, dm &\le \mathcal{E}(G_\lambda\rho, G_\lambda\rho)^{1/2} \\ &\le (\mathcal{E}(G_\lambda\rho, G_\lambda\rho) + \lambda(G_\lambda\rho, G_\lambda\rho)_{L^2})^{1/2} \qquad (1) \\ &= \left(\int_E \rho\, G_\lambda\rho\, dm\right)^{1/2}.\end{aligned}$$

This gives $\int_E \rho\, G_\lambda \rho\, dm \le 1$ and letting $\lambda \to 0$ we get $\int_E \rho\, G\rho\, dm \le 1$, in particular, the function $G\rho$ is finite m-almost everywhere.

DEFINITION 4. A bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ has the *Fatou property* if, whenever $(u_n)_{n\in I\!\!N} \in D(\mathcal{E})$ converges m-almost everywhere to $u \in L^2(E;m)$, then $\mathcal{E}(u,u) \le \liminf_n \mathcal{E}(u_n,u_n)$.

PROPOSITION 1. *If $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form, where E is m-σ-finite, then $(\mathcal{E}, D(\mathcal{E}))$ has the Fatou property.*

Proof. Let $(u_n)_{n\in I\!\!N}$ be a sequence in $D(\mathcal{E})$ that converges m-almost everywhere to u in $L^2(E;m)$; we show that $\mathcal{E}(u,u) \le \liminf_n \mathcal{E}(u_n,u_n)$.

For $\alpha > 0$, $u_n^\alpha := (-\alpha) \vee (u_n \wedge \alpha)$ is a normal contraction of u_n and so the Dirichlet property gives $\mathcal{E}(u_n^\alpha, u_n^\alpha) \le \mathcal{E}(u_n, u_n)$ [3; Chapter I, Theorem 4.12]. If we can prove $\mathcal{E}(u^\alpha, u^\alpha) \le \liminf_n \mathcal{E}(u_n^\alpha, u_n^\alpha)$, then we get the result by letting $\alpha \to \infty$ in

$$\mathcal{E}(u^\alpha, u^\alpha) \le \liminf_n \mathcal{E}(u_n^\alpha, u_n^\alpha) \le \liminf_n \mathcal{E}(u_n, u_n). \qquad (2)$$

This is because the strong convergence of u^α to u [3; Chapter I, Proposition 4.17] guarantees that $\mathcal{E}(u^\alpha, u^\alpha) \to \mathcal{E}(u,u)$ as $\alpha \to \infty$. Hence, without loss of generality, we can assume that $|u_n| \le \alpha$ for all $n \in I\!\!N$.

By Notes 1 and 2, we may assume that $\mathcal{E}(v,w) = (v - Tv, w)_{L^2}$ where T is bounded, symmetric, and has the Markov property. That is because, if $\mathcal{E}^\lambda(u,u) \le \liminf_n \mathcal{E}^\lambda(u_n,u_n)$ for all $\lambda > 0$, then we get the result by letting $\lambda \to \infty$ in

$$\mathcal{E}^\lambda(u,u) \le \liminf_n \mathcal{E}^\lambda(u_n,u_n) \le \liminf_n \mathcal{E}(u_n,u_n). \tag{3}$$

Since E is m-σ-finite, there exists $\phi \in L^2(E;m)$ where $\phi > 0$ m-almost everywhere. Let us define $\phi_n = (n\phi) \wedge 1$. The operator T can be extended from $L^2(E;m) \cap L^\infty(E;m)$ to $L^\infty(E;m)$ by setting $Tv := \lim_n T(\phi_n v)$ for $v \in L^\infty(E;m)$. The extended operator T is still positivity preserving and has the Markov property; in particular, $0 \le T1 \le 1$.

If v is a bounded function in $L^2(E;m)$, then v^2 also belongs to $L^2(E;m)$. The symmetry of T gives $(T(v^2),\phi_n)_{L^2} = (v^2, T\phi_n)_{L^2}$, and letting $n \to \infty$ shows that $T(v^2)$ is integrable and that $\int T(v^2)\,dm = \int v^2 T1\,dm$. Thus $(v - Tv, v)_{L^2}$ can be split in two as follows:

$$(v-Tv,v)_{L^2} = \frac{1}{2}\int \Big(T(v^2)-2vTv+v^2T1\Big)\,dm + \int v^2(1-T1)\,dm. \tag{4}$$

By Fatou's lemma we have $\int u^2(1-T1)\,dm \le \liminf_n \int u_n^2(1-T1)\,dm$, so we will get the result, if we can prove that

$$\int \Big(T(u^2)-2uTu+u^2T1\Big)\,dm \le \liminf_n \int \Big(T(u_n^2)-2u_nTu_n+u_n^2T1\Big)\,dm. \tag{5}$$

Suppose that $v \in L^\infty(E;m)$ and that $(v_j)_{j\in\mathbb{N}}$ is a sequence of non-negative functions with $v_j \uparrow v$ as $j \to \infty$. Since T is positivity preserving we get $\lim_j T(v_j) \le Tv$. But also, for any $n \in \mathbb{N}$, $\phi_n v_j \to \phi_n v$ in $L^2(E;m)$ as $j \to \infty$, so $T(\phi_n v) = \lim_j T(\phi_n v_j) \le \lim_j T(v_j)$. Letting $n \to \infty$ gives $Tv \le \lim_j T(v_j)$, so that $Tv = \lim_j T(v_j)$. This monotone convergence, together with positivity, shows that for every $a \in \mathbb{R}$,

$$T((u-a)^2) = \lim_j T\Big(\inf_{n\ge j}(u_n - a)^2\Big) \le \liminf_n T((u_n-a)^2). \tag{6}$$

If s is a simple function; that is, $s = \sum_{i=1}^n a_i 1_{A_i}$ where $(A_i)_{i=1}^n$ is a partition of E, and if $v \in L^\infty(E;m)$, then

$$T(v^2) - 2sTv + s^2T1 = \sum_{i=1}^n T((v-a_i)^2)1_{A_i}. \tag{7}$$

From (6) and (7), for any simple function s, we get

$$T(u^2) - 2sTu + s^2T1 \le \liminf_n \Big(T(u_n^2) - 2sTu_n + s^2T1\Big). \tag{8}$$

We also have

$$\begin{aligned}\Big|(T(u_n^2) - 2u_nTu_n + u_n^2T1) \;-\; (T(u_n^2) - 2sTu_n + s^2T1)\Big| \\ \le 2|Tu_n||u_n - s| + |u_n^2 - s^2|\,T1 \\ \le 2\alpha|u_n - s| + |u_n^2 - s^2|.\end{aligned} \tag{9}$$

Combining (8) and (9), and letting $n \to \infty$, gives

$$\begin{aligned}&T(u^2) \;-\; 2sTu + s^2T1 \\ &\quad\le \liminf_n\Big(T(u_n^2) - 2u_nTu_n + u_n^2T1\Big) + 2\alpha|u - s| + |u^2 - s^2|.\end{aligned} \tag{10}$$

Starting with (10), and then approximating u pointwise by simple functions s yields

$$T(u^2) - 2uTu + u^2T1 \le \liminf_n\Big(T(u_n^2) - 2u_nTu_n + u_n^2T1\Big). \tag{11}$$

The right hand side of (7) is a non-negative function, so setting $v = u_n$ and then approximating u_n pointwise by simple functions s, shows that $T(u_n^2) - 2u_nTu_n + u_n^2T1$ is non-negative. So by (11) and Fatou's lemma we get

$$\begin{aligned}&\int\Big(T(u^2) \;-\; 2uTu + u^2T1\Big)\,dm \\ &\quad\le \int \liminf_n\Big(T(u_n^2) - 2u_nTu_n + u_n^2T1\Big)\,dm \\ &\quad\le \liminf_n \int\Big(T(u_n^2) - 2u_nTu_n + u_n^2T1\Big)\,dm.\end{aligned} \tag{12}$$

This gives (5), and hence the result.

PROPOSITION 2. *Every positivity preserving form* $(\mathcal{E}, D(\mathcal{E}))$ *has the Fatou property.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $D(\mathcal{E})$ that converges m-almost everywhere to u in $L^2(E;m)$; we show that $\mathcal{E}(u,u) \le \liminf_n \mathcal{E}(u_n,u_n)$.

The proof consists of five observations that allow us to reduce the problem to the case of a symmetric Dirichlet form and then use the previous proposition.

1. The Fatou property depends only on the values $\mathcal{E}(v,v)$, which are the same for the symmetric positivity preserving form $(\widetilde{\mathcal{E}}, D(\mathcal{E}))$, where $\widetilde{\mathcal{E}}(v,w) = (1/2)\{\mathcal{E}(v,w) + \mathcal{E}(w,v)\}$. We may as well assume that $\mathcal{E}$ is symmetric.

2. If the bounded forms $\mathcal{E}^\lambda$ in Note 1 have the Fatou property, then the result follows by letting $\lambda \to \infty$ in $\mathcal{E}^\lambda(u,u) \le \liminf_n \mathcal{E}^\lambda(u_n,u_n) \le$

$\liminf_n \mathcal{E}(u_n, u_n)$. So we may assume that $\mathcal{E}$ is bounded, in particular, that $D(\mathcal{E}) = L^2(E; m)$.

3. Each u_n belongs to $L^2(E; m)$, so we can find a common m-σ-finite support F for $(u_n)_{n \in \mathbb{N}}$ and u. Restricting $\mathcal{E}$ to the subspace $\{v \in D(\mathcal{E}) \mid v = 0 \ m - \text{a.e. on } F^c\}$ gives a bounded, symmetric, positivity preserving form on $L^2(F; m|_F)$, without changing $\mathcal{E}(u_n, u_n)$ or $\mathcal{E}(u, u)$. Hence without loss of generality, we may assume that E itself is m-σ-finite.

4. Since E is m-σ-finite, and $\sup_n u_n^2$ is finite m-almost everywhere, for any $\beta > 0$ we can find a function ρ so that $0 < \rho \leq 1$ m-almost everywhere and $\int_E \max(\sup_n u_n^2, 1)\rho \, dm \leq \beta$. The form

$$\mathcal{E}_{\rho,\beta}(v, w) := \mathcal{E}(v, w) + \beta \int_E vw\rho \, dm, \tag{13}$$

is a bounded, symmetric, positivity preserving form and if $\mathcal{E}_{\rho,\beta}$ has the Fatou property then we get the result by letting $\beta \to 0$ in

$$\mathcal{E}(u, u) \leq \mathcal{E}_{\rho,\beta}(u, u) \leq \liminf_n \mathcal{E}_{\rho,\beta}(u_n, u_n) \leq \liminf_n \mathcal{E}(u_n, u_n) + \beta^2. \tag{14}$$

The $\mathcal{E}_{\rho,\beta}$ forms are transient, since

$$\int_E |v|\rho \, dm \leq \left(\int \rho \, dm \right)^{1/2} \left(\int v^2 \rho \, dm \right)^{1/2} \leq \mathcal{E}_{\rho,\beta}(v, v)^{1/2}, \tag{15}$$

for all $v \in D(\mathcal{E})$. So without loss of generality, we may assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient.

5. Finally, the transience allows us to transform $(\mathcal{E}, D(\mathcal{E}))$ into a Dirichlet form. From Note 3, we can find $\rho \in L^2(E; m)$ so that $G\rho < \infty$ m-almost everywhere. Let $v \in L^2(E; m)$ and note that because $\mathcal{E}$ is positivity preserving we have $\mathcal{E}(G_\lambda \rho - v \wedge G_\lambda \rho, (v - G_\lambda \rho)^+) = \mathcal{E}((v - G_\lambda \rho)^-, (v - G_\lambda \rho)^+) \leq 0$. On the other hand, since ρ is a positive function we get $\mathcal{E}(G_\lambda \rho, (v - G_\lambda \rho)^+) + \lambda(G_\lambda \rho, (v - G_\lambda \rho)^+)_{L^2} = (\rho, (v - G_\lambda \rho)^+)_{L^2} \geq 0$. These two inequalities, plus the fact that $G_\lambda \rho (v - G_\lambda \rho)^+ \leq v^2$, gives $\mathcal{E}(v \wedge G_\lambda \rho, v \wedge G_\lambda \rho) \leq \mathcal{E}(v \wedge G_\lambda \rho, v) + \lambda(v, v)_{L^2}$, and letting $\lambda \to 0$ gives

$$\mathcal{E}(v \wedge G\rho, v \wedge G\rho) \leq \mathcal{E}(v \wedge G\rho, v). \tag{16}$$

Since the operator G_λ is one-to-one [3; Chapter I, Proposition 1.5], for any set B with $0 < m(B) < \infty$ we have $G_\lambda(1_B) \geq 0$ and $G_\lambda(1_B) \neq 0$. Now $(G_\lambda \rho, 1_B)_{L^2} = (\rho, G_\lambda(1_B))_{L^2} > 0$, which shows that $G_\lambda \rho$ and hence $G\rho$ is strictly positive m-almost surely.

Let $h := G\rho$ to get a bijection between $L^2(E; h^2 m)$ and $L^2(E; m)$ given by $v \leftrightarrow vh$. Define a bilinear form on $L^2(E; h^2 m)$ by $\mathcal{E}^h(v, w) := \mathcal{E}(vh, wh)$. For any $v \in L^2(E; h^2 m)$, the inequality in (16) says that

$$\mathcal{E}^h(v \wedge 1, v \wedge 1) = \mathcal{E}(vh \wedge h, vh \wedge h) \leq \mathcal{E}(vh \wedge h, vh) = \mathcal{E}^h(v \wedge 1, v), \quad (17)$$

so that $(\mathcal{E}^h, D(\mathcal{E}^h))$ is a Dirichlet form. Now $(u_n/h) \to (u/h)$ almost everywhere with respect to $h^2\, m$, so by Proposition 1 we have

$$\mathcal{E}^h(u/h, u/h) \leq \liminf_n \mathcal{E}^h(u_n/h, u_n/h), \quad (18)$$

which is the desired result.

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Address for correspondence: Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1