Extended Dirichlet spaces

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Abstract: This note builds on the main result of [Sch] which says roughly that a Dirichlet form on $L^2(E;m)$ automatically has an extension to $L^0(E;m)$. This result has been known since the beginning of Dirichlet form theory [FOT, S] and has applications to the analysis of time changed Markov processes. However, earlier proofs of the extension used unnecessarily strong topological assumptions. Here we offer consequences of the general result, which allow us to prove well-known results of Dirichlet form theory under weakened hypotheses.

Résumé: La présente communication porte sur le résultat principal de [Sch] qui stipule essentiellement qu'une forme de Dirichlet sur $L^2(E;m)$ possède automatiquement une extension à $L^0(E;m)$. Ce résultat est connu depuis l'aube de la théorie des formes de Dirichlet [FOT, S] et possède des applications à l'analyse des processus de Markov avec changement de temps. Les preuves précédentes de l'existence d'une telle extension nécessitent toutefois des hypothèses topologiques inutilement fortes. Nous offrons ici, comme conséquence d'un résultat plus général, des preuves de quelques résultats bien connus de la théorie des formes de Dirichlet, mais sous des hypothèses affaiblies.

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Let $(E;m)$ be a measure space and $L^0(E;m)$ the vector space of real-valued measurable functions on E, where functions that agree m-almost everywhere are identified. We define a (positive semi-definite) *form* to be the pair $(\mathcal{E},\mathcal{F})$ where $\mathcal{F}$ is a linear subspace of $L^0(E;m)$ and $\mathcal{E}$ is a bilinear mapping from $\mathcal{F}\times\mathcal{F}$ to $\mathbb{R}$ such that $\mathcal{E}(u,u)\geq 0$ for every $u\in\mathcal{F}$.

Definition 1. The form $(\mathcal{E},\mathcal{F})$ satisfies the *strong sector condition* if there is a constant $K\geq 1$, so that $|\mathcal{E}(u,v)|\leq K\,\mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2}$ for all $u,v\in\mathcal{F}$.

Definition 2. The form $(\mathcal{E},\mathcal{F})$ has the *Fatou property* if $\mathcal{E}(u,u)\leq\liminf_n\mathcal{E}(u_n,u_n)$ for every $\mathcal{E}$-bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ that converges m-almost everywhere to u in $\mathcal{F}$.

L^0 Theory.

In this section we assume that $(\mathcal{E},\mathcal{F})$ satisfies the strong sector condition and has the Fatou property.

Lemma 1. *If $(u_n)_{n\in\mathbb{N}}$ is $\mathcal{E}$-Cauchy and $u_n\to 0$ m-almost everywhere, then $\mathcal{E}(u_n,u_n)\to 0$.*

Proof. As $l\to\infty$ we have $u_n-u_l\to u_n$ m-almost everywhere. Therefore the Fatou property gives $\mathcal{E}(u_n,u_n)\leq\liminf_l\mathcal{E}(u_n-u_l,u_n-u_l)$, and we get the result by letting $n\to\infty$. □

For sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, the strong sector condition gives

$$\begin{aligned}&|\mathcal{E}(u_n,v_n)-\mathcal{E}(u_l,v_l)|\\ &\quad\leq|\mathcal{E}(u_n,v_n-v_l)|+|\mathcal{E}(u_n-u_l,v_l)|\\ &\quad\leq K\,\mathcal{E}(u_n,u_n)^{1/2}\mathcal{E}(v_n-v_l,v_n-v_l)^{1/2}+K\,\mathcal{E}(u_n-u_l,u_n-u_l)^{1/2}\mathcal{E}(v_l,v_l)^{1/2}.\end{aligned}$$

If the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are $\mathcal{E}$-Cauchy, it follows that the sequence $\mathcal{E}(u_n, v_n)$ of numbers is Cauchy and so $\lim_n \mathcal{E}(u_n, v_n)$ exists. In addition, if $(u_n)_{n\in\mathbb{N}}$, $(\widehat{u}_n)_{n\in\mathbb{N}}$, and $(v_n)_{n\in\mathbb{N}}$ are $\mathcal{E}$-Cauchy sequences with $u_n - \widehat{u}_n \to 0$ m-almost everywhere as $n \to \infty$, then we have

$$|\mathcal{E}(u_n, v_n) - \mathcal{E}(\widehat{u}_n, v_n)| \leq K\,\mathcal{E}(u_n - \widehat{u}_n, u_n - \widehat{u}_n)^{1/2}\mathcal{E}(v_n, v_n)^{1/2}.$$

Lemma 1 says that the right hand side goes to zero as $n \to \infty$ so that $\lim_n \mathcal{E}(u_n, v_n) = \lim_n \mathcal{E}(\widehat{u}_n, v_n)$. Thus the extended form given below is well-defined.

Definition 3. We let the *extended form* $(\mathcal{E}, \mathcal{F}_e)$ be given by

$$\mathcal{F}_e = \{u \in L^0(E; m) \mid u_n \to u \ m\text{-a.e. for an } \mathcal{E}\text{-Cauchy sequence } (u_n)_{n\in\mathbb{N}} \in \mathcal{F}\}.$$
$$\mathcal{E}(u, v) = \lim_n \mathcal{E}(u_n, v_n).$$

Note. In connection with Definition 3, the classical work [AS] of Aronszajn and Smith should be noted.

We would like to establish that $(\mathcal{E}, \mathcal{F}_e)$ inherits some of the properties of $(\mathcal{E}, \mathcal{F})$. It is not hard to show that $\mathcal{F}_e$ is a linear subspace of $L^0(E; m)$, and that $\mathcal{E}$ is a positive semi-definite bilinear mapping from $\mathcal{F}_e \times \mathcal{F}_e$ to $\mathbb{R}$. Let $u, v \in \mathcal{F}_e$ and $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be approximating $\mathcal{E}$-Cauchy sequences in $\mathcal{F}$. The strong sector condition carries over since

$$|\mathcal{E}(u, v)| = \lim_n |\mathcal{E}(u_n, v_n)| \leq \lim_n K\,\mathcal{E}(u_n, u_n)^{1/2}\mathcal{E}(v_n, v_n)^{1/2} = K\,\mathcal{E}(u, u)^{1/2}\mathcal{E}(v, v)^{1/2}.$$

We note as well that $\mathcal{E}(u - u_n, u - u_n) = \lim_l \mathcal{E}(u_l - u_n, u_l - u_n) \to 0$ as $n \to \infty$. Combined with the strong sector condition, this implies that $\mathcal{E}(u_n, v) \to \mathcal{E}(u, v)$.

Lemma 2. *If $(u_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-bounded sequence in $\mathcal{F}$, and $u_n \to u$ m-almost everywhere, then $u \in \mathcal{F}_e$ and $\liminf_n \mathcal{E}(u_n, v) \leq \mathcal{E}(u, v) \leq \limsup_n \mathcal{E}(u_n, v)$ for every $v \in \mathcal{F}_e$. In particular, we have $\mathcal{E}(u, u) \leq \liminf_n \mathcal{E}(u_n, u_n)$.*

Proof. Since $(\mathcal{E}, \mathcal{F})$ is an inner product space, the $\mathcal{E}$-bounded sequence $(u_n)_{n\in\mathbb{N}}$ admits a subsequence, that we denote $(u_{n_i})_{i\in\mathbb{N}}$, whose Cesàro means $w_N = (1/N)\sum_{i=1}^N u_{n_i}$ are $\mathcal{E}$-Cauchy. Since $u_n \to u$ m-almost everywhere, we have $w_N \to u$ m-almost everywhere and so $u \in \mathcal{F}_e$. For $v \in \mathcal{F}_e$ we have,

$$\mathcal{E}(u, v) = \lim_N \mathcal{E}(w_N, v) = \lim_N \frac{1}{N}\sum_{i=1}^N \mathcal{E}(u_{n_i}, v),$$

and

$$\liminf_n \mathcal{E}(u_n, v) \leq \lim_N \frac{1}{N}\sum_{i=1}^N \mathcal{E}(u_{n_i}, v) \leq \limsup_n \mathcal{E}(u_n, v).$$

Positivity implies $0 \leq \mathcal{E}(u - u_n, u - u_n)$ or $\mathcal{E}(u, u_n) + \mathcal{E}(u_n, u) - \mathcal{E}(u, u) \leq \mathcal{E}(u_n, u_n)$. Taking lim sup on the left and lim inf on the right gives $\mathcal{E}(u, u) \leq \liminf_n \mathcal{E}(u_n, u_n)$. □

Lemma 3. *Suppose that each function in $\mathcal{F}$ has m-σ-finite support. Then $(\mathcal{E}, \mathcal{F}_e)$ has the Fatou property and $(\mathcal{F}_e)_e = \mathcal{F}_e$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be an $\mathcal{E}$-bounded sequence in $\mathcal{F}_e$ that converges m-almost everywhere to a function u. For each $n \in \mathbb{N}$, let $(u_{n,j})_{j\in\mathbb{N}}$ be an $\mathcal{E}$-Cauchy sequence in $\mathcal{F}$ that converges m-almost everywhere to u_n as $j \to \infty$. The set $F := \cup_{n,j}\{|u_{n,j}| > 0\}$ is m-σ-finite, and also supports the functions u_n and u. Let $(F_n)_{n\in\mathbb{N}}$ satisfy $m(F_n) < \infty$ for each $n \in \mathbb{N}$ and $F_n \uparrow F$ as $n \to \infty$.

For $n \in \mathbb{N}$, choose j_n so that $m(F_n \cap \{|u_{n,j_n} - u_n| > n^{-1}\}) \leq 2^{-n}$ and $\mathcal{E}(u_n - u_{n,j_n}, u_n - u_{n,j_n}) \leq n^{-1}$. Then $u_{n,j_n} \to u$ m-almost everywhere as $n \to \infty$ and $(u_{n,j_n})_{n\in\mathbb{N}}$ is $\mathcal{E}$-bounded. By Lemma 2, $u \in \mathcal{F}_e$ and $\mathcal{E}(u,u) \leq \liminf_n \mathcal{E}(u_{n,j_n}, u_{n,j_n}) = \liminf_n \mathcal{E}(u_n, u_n)$ which gives the Fatou property. The argument above shows that the pointwise limit of an $\mathcal{E}$-bounded sequence in $\mathcal{F}_e$ belongs to $\mathcal{F}_e$, so that $(\mathcal{F}_e)_e = \mathcal{F}_e$. □

Example 1. The conclusion $(\mathcal{F}_e)_e = \mathcal{F}_e$ is false without the σ-finiteness assumption. Let $E = \mathbb{R}$ equipped with the σ-algebra of all subsets of E, and the measure

$$m(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ +\infty & \text{otherwise.} \end{cases}$$

This way every function is measurable, and m-almost everywhere convergence coincides with pointwise everywhere convergence. Set $\mathcal{E} \equiv 0$ and $\mathcal{F}$ to be the space of continuous real-valued functions on E. Then $\mathcal{F}_e$ is the first Baire class of functions, and $(\mathcal{F}_e)_e$ is the second Baire class. But the second Baire class is strictly larger than the first; for example the indicator function $1_{\mathbb{Q}}$ belongs to $(\mathcal{F}_e)_e$, but not $\mathcal{F}_e$.

Definition 4. A form $(\mathcal{E}, \mathcal{F})$ is called *positivity preserving* if $u \in \mathcal{F}$ implies $u^+ \in \mathcal{F}$ and $\mathcal{E}(u^+, u - u^+) \geq 0$. A form $(\mathcal{E}, \mathcal{F})$ is called *Markov* if $u \in \mathcal{F}$ implies $u \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u \wedge 1, u - u \wedge 1) \geq 0$.

Note. These definitions date back at least to the seminal paper of Beurling and Deny [BD; Définition 1, Remarque 2] but with different terminology. In the L^2 case, these properties of $\mathcal{E}$ correspond to properties of the associated semigroup $(T_t)_{t\geq 0}$: a form is positivity preserving if and only if $0 \leq u$ implies $0 \leq T_t u$ [MR2; Theorem 1.5], while a form is Markov if and only if $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$ [MR1; Proposition 4.3 and Theorem 4.4].

Lemma 4. *The positivity preserving and Markov properties extend to* $(\mathcal{E}, \mathcal{F}_e)$.

Proof. Suppose that $(\mathcal{E}, \mathcal{F})$ is positivity preserving. Let $u \in \mathcal{F}_e$ and $(u_n)_{n\in\mathbb{N}} \in \mathcal{F}$ so that $\mathcal{E}(u - u_n, u - u_n) \to 0$ and $u_n \to u$ m-almost everywhere as $n \to \infty$. The positivity preserving property shows that $(u_n^+)_{n\in\mathbb{N}}$ is $\mathcal{E}$-bounded;

$$\mathcal{E}(u_n^+, u_n^+) \leq \mathcal{E}(u_n^+, u_n^+) - 2\mathcal{E}(u_n^+, u_n^-) + \mathcal{E}(u_n^-, u_n^-) = \mathcal{E}(u_n, u_n).$$

Since $u_n^+ \to u^+$ m-almost everywhere, Lemma 2 shows that $u^+ \in \mathcal{F}_e$ and

$$\mathcal{E}(u^+, u^+) \leq \liminf_n \mathcal{E}(u_n^+, u_n^+) \leq \liminf_n \mathcal{E}(u_n^+, u_n) = \liminf_n \mathcal{E}(u_n^+, u) \leq \mathcal{E}(u^+, u).$$

The proof when $(\mathcal{E}, \mathcal{F})$ has the Markov property is similar. □

Definition 5. A form $(\mathcal{E}, \mathcal{F})$ has a *square field*[1] if there is a bilinear map $\text{田} : \mathcal{F} \times \mathcal{F} \to L^1(E; m)$ such that $\text{田}(u, u) \geq 0$, $|\text{田}(u, v)| \leq K\, \text{田}(u, u)^{1/2}\, \text{田}(v, v)^{1/2}$ for $u, v \in \mathcal{F}$, and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_E \text{田}(u, v)(z)\, m(dz).$$

If $(\mathcal{E}, \mathcal{F})$ has a square field, then Lemma 1 says that $\text{田}(u_n, u_n) \to 0$ in $L^1(E; m)$ for any $\mathcal{E}$-Cauchy sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ that converges to zero m-almost everywhere as $n \to \infty$. So the arguments used for the form $\mathcal{E}$ up to Lemma 2 also apply to the square field 田. This gives the following result.

Lemma 5. *If $(\mathcal{E}, \mathcal{F})$ has a square field, then it extends uniquely to $\mathcal{F}_e$. If $(u_n)_{n \in \mathbb{N}}$ is an $\mathcal{E}$-bounded sequence in $\mathcal{F}$, and $u_n \to u$ m-almost everywhere, then $u \in \mathcal{F}_e$ and $\liminf_n \text{田}(u_n, v) \leq \text{田}(u, v) \leq \limsup_n \text{田}(u_n, v)$ for every $v \in \mathcal{F}_e$. In particular, we have $\text{田}(u, u) \leq \liminf_n \text{田}(u_n, u_n)$.*

L^2 Theory.

In this section we assume that $(\mathcal{E}, \mathcal{F})$ satisfies the strong sector condition and that $\mathcal{F} \subseteq L^2(E; m)$. We are first interested in whether or not the Fatou property holds. By way of comparison, we say that the form $(\mathcal{E}, \mathcal{F})$ is *closed* if $\mathcal{F}$ is complete with respect to the norm $\|u\|_1 = \{\mathcal{E}(u, u) + (u, u)_{L^2}\}^{1/2}$, and that $(\mathcal{E}, \mathcal{F})$ is *closable* if it has a closed extension. These properties depend on the behaviour of the associated quadratic function q,

$$q(u) = \begin{cases} \mathcal{E}(u, u) & \text{if } u \in \mathcal{F}, \\ \infty & \text{if } u \in L^2(E; m) \setminus \mathcal{F}. \end{cases}$$

It is known [D; Theorem 4.12] that the form $(\mathcal{E}, \mathcal{F})$ is closed [closable] if and only if q is L^2-lower semicontinuous on $L^2(E; m)$ [on $\mathcal{F}$].

Comparing this result with Definition 2 shows us that the Fatou property is something like an $L^0(E; m)$ version of closability. But we should be cautious with this analogy; almost everywhere convergence cannot in general be topologized, and since we do not assume that m is σ-finite, convergence in measure is not a suitable substitute.

However, since L^2 convergence implies almost everywhere convergence along a subsequence, we find that if $(\mathcal{E}, \mathcal{F})$ has the Fatou property, then it must be closable. The converse is false, as the following example demonstrates.

Example 2. Let $\mathcal{F} = L^2(E; m)$ where $E = (0, 1)$ and m is Lebesgue measure. Define a form on $\mathcal{F}$ by $\mathcal{E}(u, v) = (u, 1)_{L^2}(v, 1)_{L^2}$. Letting $u_n = (1 - n)1_{(0,1/n)} + 1_{(1/n,1)}$ so that $(u_n, 1)_{L^2} = 0$ but $u_n \to u = 1$ pointwise, we have $1 = \mathcal{E}(u, u) > \liminf_n \mathcal{E}(u_n, u_n) = 0$.

A natural question, then, is which closable forms have the Fatou property? Fatou's lemma shows that the inner product does, and a simple comparison argument shows that if $c\|u\|_{L^2}^2 \leq q(u)$ for some $c > 0$, then $(\mathcal{E}, \mathcal{F})$ has the Fatou property. In [Sch] it is shown[2]

[1] The notation 田 is based on the Chinese character Tián, which means 'field'.

[2] Under the additional, but unnecessary, assumption that $\mathcal{F}$ is dense in $L^2(E; m)$.

that if $(\mathcal{E}, \mathcal{F})$ is a closed positivity preserving form, then $q(u) \leq \liminf_n q(u_n)$ whenever $u_n \to u$ m-almost surely for $u_n, u \in L^2(E; m)$. In particular, $(\mathcal{E}, \mathcal{F})$ has the Fatou property, and $\mathcal{F} = \mathcal{F}_e \cap L^2(E; m)$.

Definition 6. If $(\mathcal{E}, \mathcal{F})$ has the Fatou property and if μ and m are mutually absolutely continuous measures, we define $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(E; \mu)$.

Lemma 6. *Let $(\mathcal{E}, \mathcal{F})$ be a closed form on $L^2(E; m)$ that has the Fatou property. If μ and m are mutually absolutely continuous, then $(\mathcal{E}, \mathcal{F}^\mu)$ is a closed form on $L^2(E; \mu)$. If $(\mathcal{E}, \mathcal{F})$ is either positivity preserving or Markov, then so is $(\mathcal{E}, \mathcal{F}^\mu)$.*

We must show that $\mathcal{F}^\mu$ is complete with respect to $\|u\|_1^\mu = \{\mathcal{E}(u, u) + (u, u)_{L^2(\mu)}\}^{1/2}$. Suppose that $(u_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{F}^\mu$ that is $\mathcal{E}$-Cauchy and also Cauchy in $L^2(E; \mu)$. Then there exists $u \in L^2(E; \mu)$ and a subsequence $(u_{n_i})_{i \in \mathbb{N}}$ that converges to u μ-almost everywhere and in $L^2(E; \mu)$. Since μ and m are mutually absolutely continuous, $u_{n_i} \to u$ m-almost everywhere, and so by Lemma 2 we see that $u \in \mathcal{F}_e$ and that $\mathcal{E}(u_{n_i} - u, u_{n_i} - u) \to 0$ as $i \to \infty$. Combined with the $L^2(E; \mu)$ convergence, this gives the closedness. Lemma 4 guarantees that $(\mathcal{E}, \mathcal{F}^\mu)$ inherits the positivity preserving or Markov property. □

For $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}^\mu)$ as above, and any measurable subset F of E, define

$$\mathcal{F}_F = \{u \in \mathcal{F} \mid u = 0 \ m\text{-almost everywhere on } F^c\},$$
$$\mathcal{F}_F^\mu = \{u \in \mathcal{F}^\mu \mid u = 0 \ \mu\text{-almost everywhere on } F^c\}.$$

Lemma 7. *Let $(\mathcal{E}, \mathcal{F})$ be a closed positivity preserving form on $L^2(E; m)$. Define a measure μ by $d\mu = g\, dm$ for a strictly positive and bounded function g, and consider the corresponding form $(\mathcal{E}, \mathcal{F}^\mu)$ on $L^2(E; \mu)$. If $(F_k)_{k \in \mathbb{N}}$ is an increasing sequence of sets in E, then $\cup_{k \in \mathbb{N}} \mathcal{F}_{F_k}$ is $\| \ \|_1$-dense in $\mathcal{F}$ if and only if $\cup_{k \in \mathbb{N}} \mathcal{F}_{F_k}^\mu$ is $\| \ \|_1^\mu$-dense in $\mathcal{F}^\mu$.*

Proof. We first show that the inclusion $\mathcal{F} \subseteq \mathcal{F}^\mu$ is dense. If $u \in \mathcal{F}^\mu = \mathcal{F}_e \cap L^2(E; \mu)$, then we can choose $u_n \in \mathcal{F}$ so that $\mathcal{E}(u_n - u, u_n - u) \to 0$ and $u_n \to u$ m-almost everywhere as $n \to \infty$. Define $v_n = (-|u|) \vee (u_n \wedge |u|)$, so that $v_n \in \mathcal{F}$, $(v_n)_{n \in \mathbb{N}}$ is $\mathcal{E}$-bounded, and $v_n \to u$ in $L^2(E; \mu)$. By taking a subsequence and then Cesàro means we obtain a sequence $(w_N)_{N \in \mathbb{N}}$ in $\mathcal{F}$ so that $\|w_N - u\|_1^\mu \to 0$ as $N \to \infty$.

Since $\cup_k \mathcal{F}_{F_k} \subseteq \cup_k \mathcal{F}_{F_k}^\mu$, it follows that if $\cup_k \mathcal{F}_{F_k}$ is $\| \ \|_1$-dense in $\mathcal{F}$, then $\cup_k \mathcal{F}_{F_k}^\mu$ is $\| \ \|_1^\mu$-dense in $\mathcal{F}^\mu$.

To prove the other direction, suppose that $\cup_{k=1}^\infty \mathcal{F}_{F_k}^\mu$ is a dense subset of $\mathcal{F}^\mu$ and let $u \in \mathcal{F}$. Since $u \in \mathcal{F}^\mu$ there is a sequence $(u_n)_{n \in \mathbb{N}}$ from $\cup_{k=1}^\infty \mathcal{F}_{F_k}^\mu$ such that $\|u_n - u\|_1^\mu \to 0$. Setting $v_n = (-|u|) \vee (u_n \wedge |u|)$, we have $v_n \in \cup_{k=1}^\infty \mathcal{F}_{F_k}$ and $(v_n)_{n \in \mathbb{N}}$ is $\mathcal{E}$-bounded. By taking a subsequence and then Cesàro means we obtain a sequence $(w_N)_{N \in \mathbb{N}}$ so that $w_N \in \cup_{k=1}^\infty \mathcal{F}_{F_k}$, and $\|w_N - u\|_1 \to 0$ as $N \to \infty$. □

Suppose that E is a topological space, and m is a σ-finite Borel measure. A sequence $(F_k)_{k \in \mathbb{N}}$ of closed sets is called an *$\mathcal{E}$-nest* whenever $\cup_{k=1}^\infty \mathcal{F}_{F_k}$ is dense in $\mathcal{F}$, and a function $u : E \to \mathbb{R}$ is called *$\mathcal{E}$-quasi-continuous* if there exists a $\mathcal{E}$-nest so that the restriction $u|_{F_k}$ is continuous for each k. We suppose that $(\mathcal{E}, \mathcal{F})$ is a closed form with the Markov property, that is, a *Dirichlet form.*

Lemma 8. *If $(\mathcal{E},\mathcal{F})$ is a Dirichlet form and $(u_n)_{n\in\mathbb{N}}$ an $\mathcal{E}$-bounded sequence of $\mathcal{E}$-quasi-continuous functions in $\mathcal{F}$ that converges everywhere to a function u, then u is $\mathcal{E}$-quasi-continuous.*

Proof. This result is proved similarly by Kuwae [K; Proposition 3.2] when E is a complete separable metric space.

Let $(u_n)_{n\in\mathbb{N}}$ and u as described above. We define an auxiliary function g on E so that $0 < g \le 1$ m-almost everywhere and $\int u_n^2\, g\, dm \le 1$ for all $n \in \mathbb{N}$. To do so, choose subsets F_N of $F := \{z \in E \mid 0 < \sup_n u_n^2(z) < \infty\}$ so that $F_N \uparrow F$ and $m(F_N) < \infty$. Define

$$g(z) := \begin{cases} 1 \wedge \left(2^N \sup_n u_n^2(z)\, m(F_N)\right)^{-1} & \text{if } z \in F_N \setminus F_{N-1},\ N = 1, 2, \dots \\ 1 & \text{if } z \notin F. \end{cases}$$

Setting $d\mu = g\, dm$, Lemma 6 shows us that $(\mathcal{E},\mathcal{F}^\mu)$ is a Dirichlet form on $L^2(E;\mu)$. Since $\sup_n \|u_n\|_1^\mu < \infty$, the usual proof [MR1; Chapter III, Proposition 3.5] shows that u is $\mathcal{E}^\mu$-quasi-continuous. Finally we note that Lemma 7 shows that every $\mathcal{E}^\mu$-quasi-continuous function is also $\mathcal{E}$-quasi-continuous. □

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