Some exceptional configurations

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Abstract

The Dirichlet form given by the intrinsic gradient on Poisson space is associated with a Markov process consisting of a countable family of interacting diffusions. By considering each diffusion as a particle with unit mass, the randomly evolving configuration can be thought of as a Radon measure valued diffusion.

The quasi-sure analysis of Dirichlet forms is used to find exceptional sets for this Markov process. We show that the process never hits certain unusual configurations, such as those with more than unit mass at some position, or those that violate the law of large numbers. Some of these results also hold for Gibbs measures with superstable interactions.

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1 Introduction

In recent work [1, 2, 3, 4, 10, 11, 12, 14, 17] the theory of Dirichlet forms has been used to construct and study Markov processes that take values in the space Γ_X of locally finite configurations on a Riemannian manifold X. The configuration space is defined by

$$\Gamma_X := \{\gamma \subset X : |\gamma \cap K| < \infty \text{ for every compact } K\}.$$

A configuration, then, is simply a collection γ of points in X with the property that only finitely many points inhabit any compact set.

A probability measure on Γ_X models a randomly chosen configuration, that is, a point process on X. The most well-known point process, the Poisson process, corresponds to the Poisson measure π_σ on Γ_X (see section 2.1 below). A diffusion process $((X_t)_{t\geq 0}, (P_\gamma)_{\gamma\in\Gamma_X})$ with values in Γ_X describes the evolution in time of a system of point processes on X, in other words, a countable family of X-diffusions.

The purpose of this paper is to examine the sample path properties of such a Γ_X-valued diffusion, and in particular, to find exceptional sets for this process, that is, subsets $N \subset \Gamma_X$ such that

$$\int_{\Gamma_X} P_\gamma \left(X_t \notin N \text{ for all } t \geq 0 \right) \mu(d\gamma) = 1,$$

for a suitable probability measure μ on Γ_X.

For simplicity, in this paper we will always take the manifold X to be the Euclidean space $\mathbb{R}^d$. The space $\Gamma_{\mathbb{R}^d}$ will be given the topology of vague convergence of measures, and measures on $\Gamma_{\mathbb{R}^d}$ are defined on the corresponding Borel sets $\mathcal{B}(\Gamma_{\mathbb{R}^d})$. We reserve the notation m for Lebesgue measure on $\mathbb{R}^d$.

Every configuration γ can be identified with the Radon measure $\sum_{x\in\gamma} \varepsilon_x$, and we will make this identification without comment. For $f \in C_0(\mathbb{R}^d)$ we let $\langle f, \gamma \rangle$ be the integral of f with respect to the measure γ, that is, $\langle f, \gamma \rangle = \sum_{x\in\gamma} f(x)$. Also the symbol $\emptyset$ will refer to the empty set or the zero measure as needed.

2 Dirichlet form

2.1 Poisson measures and Gibbs measures

In this section we describe the two families of probability measures that will serve as the invariant measure for our configuration-valued diffusion $(X_t)_{t\geq 0}$. In the free case, a Poisson measure is used to model random particles that act independently; while in the Gibbs case, a Gibbs measure is used to model random particles that interact via a potential function. Although we are mainly interested in the mathematically more challenging Gibbs case, analysis of the free case often serves as a useful guideline.

DEFINITION 1 For any non-atomic Radon measure σ on $\mathbb{R}^d$, the Poisson measure π_σ with intensity σ is the probability measure on $\Gamma_{\mathbb{R}^d}$ characterized by the formula

$$\int_{\Gamma_{\mathbb{R}^d}} \exp(\langle f, \gamma \rangle) \, \pi_\sigma(d\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{f(x)} - 1) \, \sigma(dx) \right), \tag{1}$$

for $f \in C_0(\mathbb{R}^d)$.

If A and B are disjoint Borel subsets of $\mathbb{R}^d$, then under the measure π_σ, $\gamma(A) := \langle 1_A, \gamma \rangle$ and $\gamma(B) := \langle 1_B, \gamma \rangle$ are independent Poisson random variables with means $\sigma(A)$ and $\sigma(B)$. Two other useful formulas that can be derived from equation (1) are

$$\begin{aligned} \int_{\Gamma_{\mathbb{R}^d}} \langle f, \gamma \rangle \, \pi_\sigma(d\gamma) &= \int_{\mathbb{R}^d} f(x) \, \sigma(dx), & (2) \\ \int_{\Gamma_{\mathbb{R}^d}} \left(\langle f, \gamma \rangle \langle g, \gamma \rangle - \langle fg, \gamma \rangle \right) \pi_\sigma(d\gamma) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(y) \, \sigma(dx) \sigma(dy). & (3) \end{aligned}$$

Equation (2) shows us that the intensity measure σ is the mean of the distribution π_σ.

Following [16], we now describe measures that correspond to interacting systems. These measures are examples of grand canonical Gibbs measures that appear in the classical statistical mechanics of continuous systems. More details on this class of measures can be found in [4, 15, 16].

A pair potential is any measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x)$. For a pair potential ϕ, a bounded measurable subset Λ in $\mathbb{R}^d$, and a configuration $\gamma \in \Gamma_{\mathbb{R}^d}$, the conditional energy of γ in Λ is given by the formula

$$E_\Lambda^\phi(\gamma) := \begin{cases} \sum \phi(x-y) & \text{if } \sum |\phi(x-y)| < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

where the summation is taken over all pairs $\{x, y\} \subset \gamma$ such that $\{x, y\} \cap \Lambda \neq \emptyset$. The term $\phi(x - y)$ is meant to represent the repulsive energy between a pair of particles located at x and y. Of course, when $\phi(x - y) < 0$ the particles are attracted rather than repulsed. We adopt the convention that a sum over the empty set is zero so that $E_\Lambda^\phi(\gamma) = 0$ if either $\gamma(\mathbb{R}^d) = 1$ or $\gamma(\Lambda) = 0$. We also define

$$Z_\Lambda^{z,\phi}(\gamma) := \int_{\Gamma_{\mathbb{R}^d}} \exp\left[-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda)\right] \pi_{z\sigma}(d\omega).$$

Here $\gamma_{\Lambda^c} + \omega_\Lambda$ is the configuration formed by combining the part of γ outside Λ with the part of ω inside Λ. The parameter $z > 0$ is called the activity and $\pi_{z\sigma}$ is the Poisson measure as in Definition 1. Notice that $Z_\Lambda^{z,\phi}(\gamma)$ is always strictly positive since it is greater than or equal to $\pi_{z\sigma}(\omega_\Lambda = 0) = \exp(-z\sigma(\Lambda)) > 0$.

DEFINITION 2 A probability measure μ on $\Gamma_{\mathbb{R}^d}$ is called a Gibbs measure with activity z, pair potential ϕ, and intensity measure σ if, for every bounded measurable $\Lambda \subset \mathbb{R}^d$ we have $Z_\Lambda^{z,\phi}(\gamma) < \infty$ for μ-almost every $\gamma \in \Gamma_{\mathbb{R}^d}$ and for every $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$,

$$\mu(\Delta) = \iint\limits_{\Gamma_{\mathbb{R}^d}\Gamma_{\mathbb{R}^d}} 1_\Delta(\gamma_{\Lambda^c} + \omega_\Lambda) \, \frac{\exp\left[-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda)\right]}{Z_\Lambda^{z,\phi}(\gamma)} \, \pi_{z\sigma}(d\omega)\, \mu(d\gamma). \tag{4}$$

Gibbs measures do not always exist. Apart from the obvious case when $\phi \equiv 0$, which yields the Poisson measure $\pi_{z\sigma}$, there is no guarantee that we can always find a measure satisfying the conditions of Definition 2. Looking at (4), we see that the exponential term encourages configurations of low energy and discourages those of high energy. The pair potential of a Gibbs measure must not force the particles to diverge to infinity or allow them to converge in a bounded region of space. For instance, when $z = 1$ and σ is Lebesgue measure, here are two pair potentials whose corresponding Gibbs measure fails to exist, for opposite reasons;

$$\phi(x) \equiv \infty, \qquad\qquad \phi(x) = \begin{cases} -1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

In [15, 16], Ruelle studies pair potentials that avoid these problems. He begins by defining a partition of $\mathbb{R}^d$ into cubes. For every $r = (r_1, \ldots, r_d) \in \mathbb{Z}^d$ let

$$Q_r = \left\{ x \in \mathbb{R}^d : \left(r_i - \frac{1}{2} \right) \le x_i < \left(r_i - \frac{1}{2} \right) \right\}. \tag{5}$$

DEFINITION 3

(SS) A pair potential ϕ is called superstable if there exist $A > 0$ and $B \ge 0$ so that if $\Lambda = \cup_{r \in R} Q_r$ is a finite union of cubes, then

$$E^{\phi}_{\Lambda}(\gamma_\Lambda) \ge \sum_{r \in R} \left[A\gamma(Q_r)^2 - B\gamma(Q_r) \right].$$

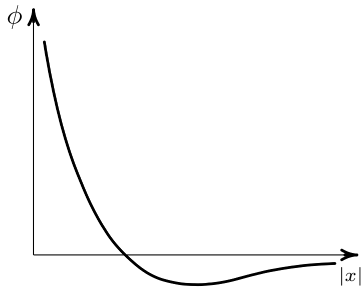
(LR) A pair potential ϕ is called lower regular if there exists a decreasing positive function $\Psi : \mathbb{N} \to [0, \infty)$ such that $\sum_{r \in \mathbb{Z}^d} \Psi(|r|_\infty) < \infty$, and for any disjoint Λ' and Λ'' that are finite unions of cubes as in (5), then we have

$$\iint\limits_{\Lambda' \Lambda''} \phi(x - y)\, \gamma(dx)\gamma(dy) \ge - \sum_{r', r'' \in \mathbb{Z}^d} \Psi(|r' - r''|_\infty) \gamma_{\Lambda'}(Q_{r'}) \gamma_{\Lambda''}(Q_{r''}),$$

for all $\gamma \in \Gamma_{\mathbb{R}^d}$. Here $|\cdot|_\infty$ refers to the maximum norm on $\mathbb{R}^d$.

(I) A pair potential ϕ is called integrable if $\int_{\mathbb{R}^d} |\exp(-\phi(x)) - 1|\, dx < \infty$.

Ruelle argues that a physically realistic model for the atoms of a rare gas, for instance, ought to use a spherically symmetric pair potential that is very repulsive at short distances and that dies out at long distances. Using Lebesgue measure as the intensity σ, he shows [16, Theorem 5.5] that if ϕ satisfies (SS), (LR), and (I), then there exists a Gibbs measure for ϕ and any $z > 0$. In fact he shows that there is always a Gibbs measure μ that is tempered in the following sense.



A typical pair potential

DEFINITION 4 A measure μ on $\Gamma_{\mathbb{R}^d}$ is called tempered if

$$\limsup_{l\to\infty} \frac{\sum_{|r|\le l} \gamma(Q_r)^2}{(2l+1)^d} < \infty \text{ for } \mu\text{-almost every } \gamma \in \Gamma_{\mathbb{R}^d}.$$

DEFINITION 5 A probability measure μ on $\Gamma_{\mathbb{R}^d}$ is called a Ruelle measure if μ is a tempered Gibbs measure with activity parameter $z > 0$, intensity σ equal to Lebesgue measure, and a pair potential ϕ that is superstable, lower regular, and integrable.

Here are a couple of examples of pair potentials that appear often in the literature and satisfy the conditions of the above definition.

1. The Lennard-Jones potential given, for some $a, b > 0$, by

   $$\phi_{a,b}(x) := \frac{a}{|x|^{12}} - \frac{b}{|x|^6}, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

   is especially important in atomic and molecular physics.

2. The potential for the hard-core model is given, for some radius $R > 0$, by

   $$\phi_R(x) = \begin{cases} +\infty & \text{if } |x| \le R, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that μ is a Gibbs measure and Λ a bounded measurable subset of $\mathbb{R}^d$. Let $\mathcal{F}(\Lambda)$ be the σ-algebra of events $\Delta \in \mathcal{B}(\Gamma_{\mathbb{R}^d})$ that only depend on the part of the configuration in Λ, that is, $1_\Delta(\gamma) = 1_\Delta(\gamma_\Lambda)$ for every $\gamma \in \Gamma_{\mathbb{R}^d}$. Exchanging the order of integration in (4), we find that $\mu|_{\mathcal{F}(\Lambda)}$ is absolutely continuous with respect to $\pi_{z\sigma}|_{\mathcal{F}(\Lambda)}$ with density

$$\omega \mapsto \int_{\Gamma_{\mathbb{R}^d}} [Z_\Lambda^{z,\phi}(\gamma)]^{-1} \exp\left[-E_\Lambda^\phi(\gamma_{\Lambda^c} + \omega_\Lambda)\right] \mu(d\gamma). \tag{6}$$

In other words, a Gibbs measure is always locally absolutely continuous with respect to its corresponding Poisson measure. In general we have very little information about the density (6), but for Ruelle measures it is known to be bounded, with a bound that depends on Λ.

Another important tool in studying Ruelle measures are the (infinite-volume) correlation functions $\rho_m : (\mathbb{R}^d)^m \to \mathbb{R}$ which can be expressed as

$$\begin{aligned} &\rho_m(x_1, \ldots, x_m) \\ &\quad = z^m \exp(-E_\Lambda^\phi(\{x_1, \ldots, x_m\})) \int_{\Gamma_{\mathbb{R}^d}} \exp\Big(- \sum_{i=1}^m \langle \phi(x_i - \cdot), \gamma \rangle \Big) \mu(d\gamma). \end{aligned}$$

These provide us with the analogues of (2) and (3) for Ruelle measures:

$$\begin{aligned} \int_{\Gamma_{\mathbb{R}^d}} \langle f, \gamma \rangle \, \mu(d\gamma) &= \int_{\mathbb{R}^d} f(x) \rho_1(x) \, dx, && (7) \\ \int_{\Gamma_{\mathbb{R}^d}} \left(\langle f, \gamma \rangle \langle g, \gamma \rangle - \langle fg, \gamma \rangle \right) \mu(d\gamma) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)\, g(y) \rho_2(x, y) \, dx \, dy. && (8) \end{aligned}$$

2.2 Gradient

We define a linear space $\mathcal{F}C_b^\infty$ of functions on $\Gamma_{\mathbb{R}^d}$ by taking all smooth cylinder functions. That is,

$$\begin{aligned} \mathcal{F}C_b^\infty := \{\, u : u(\gamma) = {} & g(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \\ & \text{for some } f_i \in C_0^\infty(\mathbb{R}^d) \text{ and } g \in C_b^\infty(\mathbb{R}^n) \,\}. \end{aligned}$$

For $u \in \mathcal{F}C_b^\infty$, we define the gradient $\nabla^\Gamma u$ at the point $\gamma \in \Gamma_{\mathbb{R}^d}$ as an element of the "tangent space" $T_\gamma(\Gamma_{\mathbb{R}^d}) := L^2(\mathbb{R}^d \to \mathbb{R}^d; \gamma)$ by the formula

$$\left(\nabla^\Gamma u \right)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i} \left(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle \right) \nabla f_i(x).$$

Here ∇ refers to the usual gradient on $\mathbb{R}^d$. It is not hard to prove that $\nabla^\Gamma u$ is well-defined, even though the representation of u as a cylinder function is not unique.

2.3 Dirichlet form

DEFINITION 6 For $u, v \in \mathcal{F}C_b^\infty$ define the square field $\text{田}(u, v)$[1] as the real-valued function on $\Gamma_{\mathbb{R}^d}$ given by

$$\begin{aligned} \text{田}(u, v)(\gamma) &:= \langle \nabla^\Gamma u, \nabla^\Gamma v \rangle_{T_\gamma(\Gamma_{\mathbb{R}^d})} \\ &= \int_{\mathbb{R}^d} \langle (\nabla^\Gamma u)(\gamma; x), (\nabla^\Gamma v)(\gamma; x) \rangle_{\mathbb{R}^d} \, \gamma(dx). \end{aligned}$$

We will often use the abbreviation $\text{田}(u) := \text{田}(u, u)$.

DEFINITION 7 For $u, v \in \mathcal{F}C_b^\infty$ define the pre-Dirichlet form by

$$\mathcal{E}(u, v) := \int_{\Gamma_{\mathbb{R}^d}} \text{田}(u, v)(\gamma) \, \mu(d\gamma). \qquad (9)$$

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[1] This notation for square field is based on the Chinese character Tián, which means 'field'.

We would like to study the form in (9), where μ is a Gibbs measure or a mixture of Poisson measures. A basic requirement is that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ should be a well-defined and closable bilinear form on $L^2(\Gamma_{\mathbb{R}^d}, \mu)$. In general, proving closability is difficult and the closability problem is still an active area of research. In the free case, the specific conditions in Definition 8 (1.) guarantee closability, but in the Ruelle case we will simply assume that the pair potential ϕ satisfies the (mild) additional smoothness and integrability assumptions to ensure that (9) is well-defined and closable. In particular, the Lennard-Jones and hard-core potentials yield closable forms. Proofs of closability can be found in [3, Corollary 4.1 and Remark 4.3] for the free case, and in [4, Proposition 5.1], [11, Section 6.3], and [10] in the Ruelle case.

DEFINITION 8

1. Let σ be a measure on $\mathbb{R}^d$ that has a density ϱ with respect to Lebesgue measure satisfying $\varrho > 0$ almost everywhere, and $\varrho^{1/2} \in H^{1,2}_{\text{loc}}(\mathbb{R}^d)$. Here $H^{1,2}_{\text{loc}}(\mathbb{R}^d)$ denotes the local Sobolev space of order 1 in $L^2_{\text{loc}}(\mathbb{R}^d; dx)$. We are in the free case when μ is a mixture of $\pi_{z\sigma}$, that is,
$$\mu := \int_{\mathbb{R}_+} \pi_{z\sigma}\, \lambda(dz), \tag{10}$$
where λ is a probability measure on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} z\, \lambda(dz) < \infty$.
2. We are in the Ruelle case when μ is a Ruelle measure such that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is a well-defined and closable form in $L^2(\Gamma_{\mathbb{R}^d}, \mu)$.

From now on, we assume that the measure μ is in one of the two categories described in Definition 8, and we will let $(\mathcal{E}, D(\mathcal{E}))$ denote the closure of $(\mathcal{E}, \mathcal{F}C_b^\infty)$ in $L^2(\Gamma_{\mathbb{R}^d}, \mu)$. Standard Dirichlet form theory shows that the closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric, local, Dirichlet form. In particular, the space $D(\mathcal{E})$ is complete with respect to the norm
$$|u|_1 := \left(\mathcal{E}(u,u) + (u,u)_{L^2(\mu)}\right)^{1/2}.$$
The map $(u, v) \to \boxplus(u, v)$ is continuous from $\mathcal{F}C_b^\infty \times \mathcal{F}C_b^\infty$ into $L^1(\Gamma_{\mathbb{R}^d}; \mu)$ when $\mathcal{F}C_b^\infty$ is equipped with the $|\ |_1$-norm, and so the square field $\boxplus$ extends to the full domain $D(\mathcal{E})$ in such a way that formula (9) continues to hold.

The usual functional calculus for Dirichlet forms ensures (eg. [13, Lemma 3.2]) that if $u, v \in D(\mathcal{E})$ and ψ is a smooth function on $\mathbb{R}$ that vanishes at the origin and has bounded derivative, then $\psi(u)$ belongs to $D(\mathcal{E})$ and
$$\boxplus(\psi(u)) = (\psi'(u))^2 \boxplus(u). \tag{11}$$
In the same vein, you can show that $u \vee v$ and $u \wedge v$ belong to $D(\mathcal{E})$ and
$$\boxplus(u \vee v) \leq \boxplus(u) \vee \boxplus(v) \quad \text{and} \quad \boxplus(u \wedge v) \leq \boxplus(u) \vee \boxplus(v). \tag{12}$$

Let us do a sample calculation with the easiest kind of cylinder function: $u(\gamma) = \langle f, \gamma \rangle$ where $f \in C_0^\infty(\mathbb{R}^d)$. Although u doesn't belong to $\mathcal{F}C_b^\infty$, it is easy to show that it belongs to $D(\mathcal{E})$ and that the following calculations are valid. For this function u, the gradient $\nabla^\Gamma u$ is equal to ∇f at all points $\gamma \in \Gamma_{\mathbb{R}^d}$, and so the square field is $\boxplus(u)(\gamma) = \int_{\mathbb{R}^d} |\nabla f|^2(x)\gamma(dx)$. In the free case, it follows from equation (2) and (10) that the Dirichlet form applied to such a function u gives

$$\mathcal{E}(u,u) = \int_{\Gamma_{\mathbb{R}^d}} \langle |\nabla f|^2, \gamma \rangle \, \mu(d\gamma) = \Big(\int_{\mathbb{R}_+} z \, \lambda(dz) \Big) \Big(\int_{\mathbb{R}^d} |\nabla f|^2(x) \sigma(dx) \Big),$$

In the Ruelle case, (7) shows us that

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} |\nabla f|^2(x) \, \rho_1(x) \, dx.$$

2.4 Stochastic Process

In order to prove that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ has an associated Markov process we need to show that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular [7, Chapter IV, Theorem 3.5]. The quasi-regularity and locality of $(\mathcal{E}, D(\mathcal{E}))$ has been proven for certain cases by Yoshida [17], and in general by Ma and Röckner [8] but since $\Gamma_{\mathbb{R}^d}$ is not complete with respect to the vague topology it is necessary to use the completed state space

$$\ddot{\Gamma}_{\mathbb{R}^d} := \{\mathbb{Z}_+ \cup \{+\infty\}\text{-valued Radon measures on } \mathbb{R}^d\}.$$

Since $\Gamma_{\mathbb{R}^d} \subset \ddot{\Gamma}_{\mathbb{R}^d}$ and $\mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d}) \cap \Gamma_{\mathbb{R}^d} = \mathcal{B}(\Gamma_{\mathbb{R}^d})$, we can consider μ as a measure on $(\ddot{\Gamma}_{\mathbb{R}^d}, \mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma}_{\mathbb{R}^d}; \mu)$.

The associated Markov process $((X_t)_{t \geq 0}, (P_\gamma)_{\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}})$ has vaguely continuous sample paths since $(\mathcal{E}, D(\mathcal{E}))$ is a local form [7, Chapter V, Theorem 1.11].

2.5 Exceptional Sets

Let $(\mathcal{E}, D(\mathcal{E}))$ be the local, quasi-regular Dirichlet form given by the closure of the pre-Dirichlet form in (9), and $(X_t)_{t \geq 0}$ the associated $\ddot{\Gamma}_{\mathbb{R}^d}$-valued diffusion process.

Exceptional sets and quasi-continuous functions are important tools for understanding the diffusion process corresponding to a Dirichlet form. Exceptional sets are "almost empty", and quasi-continuous functions are "almost continuous" in a sense appropriate for Dirichlet forms. We first give the definition of these objects in terms of the form $(\mathcal{E}, D(\mathcal{E}))$, and then describe their interpretation using the process $(X_t)_{t \geq 0}$.

DEFINITION 9 Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(\ddot{\Gamma}_{\mathbb{R}^d}; \mu)$.

(a) For a closed subset $F \subseteq \ddot{\Gamma}_{\mathbb{R}^d}$ we define a closed subspace of $D(\mathcal{E})$ by

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) : u = 0 \ \ \mu\text{-a.e. on } \ \ddot{\Gamma}_{\mathbb{R}^d} \setminus F\}.$$

(b) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of $\ddot{\Gamma}_{\mathbb{R}^d}$ is called an $\mathcal{E}$-nest if $\cup_{k\geq 1} D(\mathcal{E})_{F_k}$ is $| \ |_1$-dense in $D(\mathcal{E})$.

(c) A function $u : \ddot{\Gamma}_{\mathbb{R}^d} \to \mathbb{R}$ is called $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ so that $u|_{F_k}$ is continuous for each $k \in \mathbb{N}$.

(d) A subset $N \in \mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d})$ is called $\mathcal{E}$-exceptional if 1_N is $\mathcal{E}$-quasi-continuous and $\mu(N) = 0$.

We shall use the following result throughout this paper (see [7, Chapter III, Proposition 3.5], [7, Chapter IV, Lemma 4.5], and [6]).

LEMMA 1 Let $u_n \in D(\mathcal{E})$ be a sequence of $\mathcal{E}$-quasi-continuous functions with $\sup_n \mathcal{E}(u_n, u_n) < \infty$ and $u_n \to u$ pointwise. Then u is an $\mathcal{E}$-quasi-continuous function, in particular, for μ-almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$,

$$P_\gamma \left(t \to u(X_t) \text{ is continuous }\right) = 1. \tag{13}$$

If u is μ-square integrable, then $u \in D(\mathcal{E})$.

It is also worthwhile noting the interpretation of an $\mathcal{E}$-exceptional set in terms of the associated diffusion [7, Chapter IV, Proposition 5.30].

LEMMA 2 A set $N \in \mathcal{B}(\ddot{\Gamma}_{\mathbb{R}^d})$ is $\mathcal{E}$-exceptional if and only if, for μ-almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$,

$$P_\gamma \left(X_t \in N \text{ for some } 0 \leq t < \infty\right) = 0.$$

3 Support properties for the process

Throughout this section, we will assume that $(\mathcal{E}, D(\mathcal{E}))$ is the closure of the pre-Dirichlet form (9) and that μ is in one of the two categories of measures spelled out in Definition 8. From the results in [8] we know that $(\mathcal{E}, D(\mathcal{E}))$ is a local, quasi-regular Dirichlet form on the space $L^2(\ddot{\Gamma}_{\mathbb{R}^d}, \mu)$. We are interested in the sample path properties of the associated $\ddot{\Gamma}_{\mathbb{R}^d}$-valued diffusion process X_t.

3.1 Local properties

The space $\ddot{\Gamma}_{\mathbb{R}^d}$ is the completion of the space $\Gamma_{\mathbb{R}^d}$ in the vague topology. Unlike the measures $\sum_{x\in\gamma}\varepsilon_x$ in the space $\Gamma_{\mathbb{R}^d}$, the measures in $\ddot{\Gamma}_{\mathbb{R}^d}$ allow for the possibility that two (or more) particles could occupy the same position in $\mathbb{R}^d$, resulting in a point with mass of two (or more). The following proposition gives conditions so that, with probability one, the process X_t will not hit the set of such measures, so the completion of the state space was unnecessary after all.

The results in the first two propositions in this section have been announced in [14].

PROPOSITION 1 (Free case) If $d \geq 2$, $\varrho \in L^2_{\text{loc}}(dx)$, and $\int_{\mathbb{R}_+} z^2\,\lambda(dz) < \infty$, then the set $\ddot{\Gamma}_{\mathbb{R}^d} \setminus \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$-exceptional.

Proof. Our goal is to show that the set of measures γ that take values greater than one is $\mathcal{E}$-exceptional. It clearly suffices to prove this locally, that is, to show that for every positive integer a, the function $u := 1_N$ is $\mathcal{E}$-quasi-continuous, where

$$N := \{\gamma : \sup(\gamma(\{x\}) : x \in [-a,a]^d) \geq 2\}. \tag{14}$$

We first note that if μ is a mixed Poisson measure, then $\sup_x \gamma(\{x\}) = 1$ μ-almost everywhere, and so $\mu(N) = 0$.

Our analysis begins with a smooth partition of $\mathbb{R}^d$ into small pieces. Let f be a $C_0^\infty(\mathbb{R})$ function satisfying $1_{[0,1]} \leq f \leq 1_{[-1/2,3/2)}$ and $|f'| \leq 3 \times 1_{[-1/2,3/2)}$, and for any $n \in \mathbb{N}$ and $i = (i_1,\ldots,i_d) \in \mathbb{Z}^d$, define a $C_0^\infty(\mathbb{R}^d)$ function by

$$f_i(x) := \prod_{k=1}^d f(nx_k - i_k).$$

We also let $I_i(x) := \prod_{k=1}^d 1_{[-1/2,3/2)}(nx_k - i_k)$ and note that $f_i \leq I_i$. Taking the jth partial derivative of f_i gives

$$\partial_j f_i(x) = nf'(nx_j - i_j)\prod_{k\neq j} f(nx_k - i_k),$$

and so $(\partial_j f_i(x))^2 \leq 9n^2 I_i(x)$. Adding over j from 1 to d gives us

$$|\nabla f_i(x)|^2 \leq 9n^2 d I_i(x). \tag{15}$$

Let ψ be a smooth function on $\mathbb{R}$ satisfying $1_{[2,\infty)} \leq \psi \leq 1_{[1,\infty)}$ and $|\psi'| \leq 2 \times 1_{(1,\infty)}$. Choosing $A := \mathbb{Z}^d \cap [-na,na]^d$, define a continuous element of $D(\mathcal{E})$ by

$$u_n(\gamma) := \psi\big(\sup_{i\in A}\langle f_i,\gamma\rangle\big). \tag{16}$$

Then $u_n \to u$ pointwise as $n \to \infty$, so to apply Lemma 1 we must prove that $\sup_n \mathcal{E}(u_n, u_n) < \infty$. We begin by bounding $\boxplus(u_n)$, the square field applied to u_n. First note that

$$\Big(\psi'\big(\sup_{i\in A}\langle f_i,\gamma\rangle\big)\Big)^2 \le 4 \times 1_{(\sup_{i\in A}\langle f_i,\gamma\rangle > 1)} \le 4 \times 1_{(\sup_{i\in A}\langle I_i,\gamma\rangle \ge 2)}, \tag{17}$$

where for the final inequality we use the fact that $\langle I_i, \gamma\rangle$ is an integer. Therefore, using (12) along with the inequalities in (15) and (17), we get

$$\begin{aligned}
\boxplus(u_n)(\gamma) &= \Big(\psi'\big(\sup_{i\in A}\langle f_i,\gamma\rangle\big)\Big)^2 \boxplus\big(\sup_{i\in A}\langle f_i,\cdot\rangle\big)(\gamma) \\
&\le \Big(\psi'\big(\sup_{i\in A}\langle f_i,\gamma\rangle\big)\Big)^2 \sup_{i\in A}\boxplus(\langle f_i,\cdot\rangle)(\gamma) \\
&= \Big(\psi'\big(\sup_{i\in A}\langle f_i,\gamma\rangle\big)\Big)^2 \sup_{i\in A}\int |\nabla f_i(x)|^2\,\gamma(dx) \\
&\le 4 \times 1_{(\sup_{i\in A}\langle I_i,\gamma\rangle \ge 2)}\, 9\, n^2\, d\, \sup_{i\in A}\langle I_i,\gamma\rangle \\
&\le 36 n^2 d \sum_{i\in A} 1_{(\langle I_i,\gamma\rangle \ge 2)}\langle I_i,\gamma\rangle.
\end{aligned} \tag{18}$$

From equation (10) we get

$$\begin{aligned}
\int_{(\langle I_i,\gamma\rangle \ge 2)}\langle I_i,\gamma\rangle \mu(d\gamma) &= \int_{\mathbb{R}_+} z\langle I_i,\sigma\rangle \left(1 - e^{-z\langle I_i,\sigma\rangle}\right)\,\lambda(dz) \\
&\le \langle I_i,\sigma\rangle^2 \int_{\mathbb{R}_+} z^2\,\lambda(dz)
\end{aligned} \tag{19}$$

and combined with (18) this gives

$$\mathcal{E}(u_n, u_n) \le c\, n^2 \sum_{i\in A}\langle I_i,\sigma\rangle^2. \tag{20}$$

Although the supports of the indicator functions I_i are not disjoint, each point belongs to at most 2^d of the sets $\{I_i = 1\}$ for $i \in A$. Therefore the Cauchy-Schwarz inequality gives us

$$\begin{aligned}
\sum_{i\in A}\langle I_i,\sigma\rangle^2 &= \sum_{i\in A}\left(\int I_i(x)\,\varrho(x)\,dx\right)^2 \\
&\le \sum_{i\in A}\left(\int I_i(x)\,\varrho(x)^2\,dx\right)\left(\int I_i(x)\,dx\right) \\
&\le 2^d \int_{[-(a+1),a+1]^d} \varrho(x)^2\,dx\,(2/n)^d,
\end{aligned} \tag{21}$$

and combining this with (20) we find that

$$\mathcal{E}(u_n, u_n) \le cn^{2-d}.$$

Since we have assumed that $d \ge 2$ we see that $\sup_n \mathcal{E}(u_n, u_n) < \infty$. We conclude that N is $\mathcal{E}$-exceptional. □

Note Corollaries 1 and 2 in the next section tell us when we can drop the condition $\int_{\mathbb{R}_+} z^2\,dz < \infty$.

PROPOSITION 2 (Ruelle case) The set $\ddot{\Gamma}_{\mathbb{R}^d} \setminus \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$-exceptional.

Proof. To compare with the free case we first set $\varrho = z$ so that σ is z times Lebesgue measure, and put $\lambda = \varepsilon_z$. The inequality (5.14) in [16] guarantees that the Radon-Nikodym derivative in (6) is bounded, for any bounded Λ. Putting $\Lambda = [-a, a]^d$ and $A = \mathbb{Z}^d \cap [-na, na]^d$ we see that the inequalities (19), (20), and (21) are valid (up to a constant) for a Ruelle measure. Therefore, the function u is $\mathcal{E}$-quasi-continuous and $\mu(N) = 0$ in the Ruelle case as well. $\square$

3.2 Global properties

In this section we consider properties of the random configuration X_t that depend on the whole configuration. Here, the local absolute continuity of a Ruelle measure with respect to Poisson measure is not helpful, so it is harder to get results for Ruelle measures.

The next result shows that, in the free case, under the exponential growth condition in (22), the total number of particles $X_t(\mathbb{R}^d)$ in the random configuration X_t remains constant in time. Note that (22) is trivially satisfied if $\sigma(\mathbb{R}^d) < \infty$, and that in this case $X_t(\mathbb{R}^d) < \infty$ also. In all of the calculations below, c is a constant that does not depend on the index n, but whose value may change from line to line. Also, B_r refers to the ball in $\mathbb{R}^d$ with radius r, centered at the origin.

PROPOSITION 3 (Free case) If there exist $a, b > 0$ so that

$$\sigma(B_r) \leq a \exp(br), \tag{22}$$

then

$$\int_{\ddot{\Gamma}_{\mathbb{R}^d}} P_\gamma \left(t \to X_t(\mathbb{R}^d) \text{ is constant } \right) \mu(d\gamma) = 1.$$

Proof. Let ψ_k be a smooth function on $\mathbb{R}$ that vanishes outside of the interval $(k-1, k+1)$ and so that $\psi_k(k) = 1$. Also let f_r be a smooth function on $\mathbb{R}^d$ such that $1_{B_r} \leq f_r \leq 1_{B_{r+1}}$ and $|\nabla f_r|^2 \leq c$. Let

$$u_r(\gamma) := \psi_k \left(\langle f_r, \gamma \rangle \right). \tag{23}$$

In order to estimate $\mathcal{E}(u_r, u_r)$ we first note that

$$\nabla^\Gamma u_r(\gamma; x) = \psi_k' \left(\langle f_r, \gamma \rangle \right) \nabla f_r(x),$$

so that the square field satisfies

$$\begin{aligned} \boxplus(u_r)(\gamma) &\leq \left(\psi_k' \left(\langle f_r, \gamma \rangle \right)\right)^2 \textstyle\int_{\mathbb{R}^d} |\nabla f_r(x)|^2 \, \gamma(dx) \\ &\leq c 1_{\{\gamma(B_r) \leq k\}} \gamma(B_{r+1} \setminus B_r). \end{aligned} \tag{24}$$

Therefore it follows that

$$\begin{aligned}
\mathcal{E}(u_r, u_r) &\leq c \int_{\mathbb{R}_+} \int_{(\gamma(B_r)\leq k)} \gamma(B_{r+1} \setminus B_r)\, \pi_{z\sigma}(d\gamma)\, \lambda(dz) \\
&\leq c \int_{\mathbb{R}_+} \left[\sum_{j=0}^{k} e^{-z\sigma(B_r)} z^j \sigma(B_r)^j \right] z\sigma(B_{r+1} \setminus B_r)\, \lambda(dz) \\
&\leq c \int_{\mathbb{R}_+} \left[\sum_{j=1}^{k+1} e^{-z\sigma(B_r)} z^j \sigma(B_r)^j \right] \sigma(B_{r+1})/\sigma(B_r)\, \lambda(dz) \\
&\leq c\, \sigma(B_{r+1})/\sigma(B_r).
\end{aligned}$$

We used the fact that B_r and $B_{r+1} \setminus B_r$ are disjoint so the random variables $\gamma(B_r)$ and $\gamma(B_{r+1} \setminus B_r)$ are independent on $(\ddot{\Gamma}_{\mathbb{R}^d}, \pi_{z\sigma})$. By the assumption in (22), we can find a sequence $r_n \to \infty$ so that $\sup_n \sigma(B_{r_n+1})/\sigma(B_{r_n}) < \infty$ and therefore $\sup_n \mathcal{E}(u_{r_n}, u_{r_n}) < \infty$. As $n \to \infty$ the sequence $(u_{r_n})_{n\in\mathbb{N}}$ converges pointwise to the function $\psi_k(\gamma(\mathbb{R}^d)) = 1_{\{\gamma(\mathbb{R}^d)=k\}}$. By Lemma 1, we find that

$$\int_{\ddot{\Gamma}_{\mathbb{R}^d}} P_\gamma \left(t \to 1_{\{\gamma(\mathbb{R}^d)=k\}}(X_t) \text{ is continuous } \right) \mu(d\gamma) = 1,$$

and since k is arbitrary, we obtain our result. □

COROLLARY 1 The condition $\int_{\mathbb{R}_+} z^2\, \lambda(dz) < \infty$ in Proposition 1 can be dropped if $\sigma(\mathbb{R}^d) < \infty$.

Proof. The proof depends on a truncation argument based on the fact that if $u, v \in D(\mathcal{E})$ are bounded functions, then the product uv belongs to $D(\mathcal{E})$ and

$$\text{田}(uv) = u^2\, \text{田}(v) + v^2\, \text{田}(u). \tag{25}$$

If $\sigma(\mathbb{R}^d) < \infty$, then the mixed measure μ is absolutely continuous with respect to π_σ and the density is given by

$$\frac{d\mu}{d\pi_\sigma}(\gamma) = \int_{\mathbb{R}_+} e^{(1-z)\sigma(\mathbb{R}^d)} z^{\gamma(\mathbb{R}^d)}\, \lambda(dz).$$

In particular, $\gamma(\mathbb{R}^d) < \infty$ for μ-almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$ and μ is just a multiple of π_σ on each of the sets $\{\gamma : \gamma(\mathbb{R}^d) = k\}$. In Proposition 3 we showed that $v(\gamma) := 1_{\{\gamma(\mathbb{R}^d)=k\}}$ is an $\mathcal{E}$-quasi-continuous element of $D(\mathcal{E})$ and by letting $r \to \infty$ in (24) we see that $\text{田}(v) = 0$. In particular, (25) shows that for any bounded $u \in D(\mathcal{E})$, we have $u 1_{\{\gamma(\mathbb{R}^d)=k\}} \in D(\mathcal{E})$ and $\text{田}(u 1_{\{\gamma(\mathbb{R}^d)=k\}}) = 1_{\{\gamma(\mathbb{R}^d)=k\}} \text{田}(u)$.

Now apply this formula to the functions $(u_n)_{n\in\mathbb{N}}$ defined in (16). Using the density $d\mu/d\pi_\sigma$ and (18), we find that

$$\begin{aligned}
\int_{\ddot{\Gamma}_{\mathbb{R}^d}} \boxplus(u_n 1_{\{\gamma(\mathbb{R}^d)=k\}})(\gamma)\,\mu(d\gamma) &= \int_{\{\gamma(\mathbb{R}^d)=k\}} \boxplus(u_n)(\gamma)\,\mu(d\gamma) \\
&= c\int_{\{\gamma(\mathbb{R}^d)=k\}} \boxplus(u_n)(\gamma)\,\pi_\sigma(d\gamma) \\
&\leq cn^2 \sum_{i\in A} \langle I_i, \sigma\rangle^2.
\end{aligned}$$

As in the proof of Proposition 1, since $d \geq 2$, the expression above is bounded in n and so we conclude that $u1_{\{\gamma(\mathbb{R}^d)=k\}}$ is an $\mathcal{E}$-quasi-continuous element of $D(\mathcal{E})$, where u is the limit of the sequence $(u_n)_{n\in\mathbb{N}}$. This means that the set

$$\{\gamma : \gamma(\mathbb{R}^d) = k\} \cap N$$

is $\mathcal{E}$-exceptional, where N is defined in (14). This gives the result, since k and a are arbitrary, and since Proposition 3 shows that $\{\gamma : \gamma(\mathbb{R}^d) = \infty\}$ is $\mathcal{E}$-exceptional. □

We would now like to prove the result corresponding to Proposition 3 in the Ruelle case, that is, show that $X_t(\mathbb{R}^d) = \infty$ for all times $t \geq 0$. We begin by proving the fixed time result which says that $\mu(\gamma(\mathbb{R}^d) < \infty) = 0$. Here μ need not even be a Ruelle measure, any Gibbs measure with intensity satisfying $\sigma(\mathbb{R}^d) = \infty$ works.

LEMMA 3 A Gibbs measure μ with $\sigma(\mathbb{R}^d) = \infty$ does not charge the set of finite configurations.

Proof. First consider the case where $Z^\phi_\Lambda(\emptyset) < \infty$ for all bounded Λ. Then

$$\begin{aligned}
\mu(\{\emptyset\}) &= \int \frac{1}{Z^\phi_\Lambda(\gamma)} \int_{\{\omega_\Lambda+\gamma_{\Lambda^c}=\emptyset\}} \exp\left[-E^\phi_\Lambda(\omega_\Lambda + \gamma_{\Lambda^c})\right]\,\pi_{z\sigma}(d\omega)\,\mu(d\gamma) \\
&= \int_{\{\gamma_{\Lambda^c}=\emptyset\}} \frac{1}{Z^\phi_\Lambda(\gamma)} \exp\left[-z\sigma(\Lambda)\right]\,\mu(d\gamma) \qquad (26) \\
&= \frac{1}{Z^\phi_\Lambda(\emptyset)} \exp\left[-z\sigma(\Lambda)\right]\mu(\gamma_{\Lambda^c} = 0).
\end{aligned}$$

Now $Z^\phi_\Lambda(\emptyset) \geq \exp[-z\sigma(\Lambda)](1 + z\sigma(\Lambda))$, because $E^\phi_\Lambda(\omega_\Lambda) = 0$ if $\omega(\Lambda)$ is either 0 or 1. Combined with (26) this yields $\mu(\{\emptyset\}) \leq (1+z\sigma(\Lambda))^{-1}$ which goes to zero as $\Lambda \nearrow \mathbb{R}^d$. Plugging this back into the equation (26) gives $\mu(\gamma_{\Lambda^c} = 0) = 0$ for all Λ, which in turn implies that $\mu(\gamma(\mathbb{R}^d) < \infty) = 0$.

On the other hand, if $Z^\phi_\Lambda(\emptyset) = \infty$ for large Λ, then setting Δ in (4) to be the set of finite configurations, and bearing in mind that $\gamma_\Lambda(\mathbb{R}^d) < \infty$ and

$\omega_\Lambda(\mathbb{R}^d) < \infty$ anyway, we get

$$\mu(\gamma(\mathbb{R}^d) < \infty) = \iint\limits_{\{\gamma(\mathbb{R}^d)<\infty\}} \frac{1}{Z^\phi_\Lambda(\gamma)} \exp\left[-E^\phi_\Lambda(\omega_\Lambda + \gamma_{\Lambda^c})\right] \pi_{z\sigma}(d\omega)\,\mu(d\gamma).$$

However, if $\gamma(\mathbb{R}^d) < \infty$, then $Z^\phi_\Lambda(\gamma) \to Z^\phi_\Lambda(\emptyset) = \infty$ as $\Lambda \nearrow \mathbb{R}^d$ and by bounded convergence we obtain $\mu(\gamma(\mathbb{R}^d) < \infty) = 0$. □

PROPOSITION 4 (Ruelle case) For sufficiently small activity z, the set $\{\gamma : \gamma(\mathbb{R}^d) < \infty\}$ is $\mathcal{E}$-exceptional.

Proof. Mimicking the proof of Proposition 3, we define u_r as in (23). Use Cauchy-Schwarz with q defined by $1/q + 1/d = 1$ to integrate the bound from (24) and obtain

$$\mathcal{E}(u_r, u_r) \leq c\,\mu\left(\gamma(B_r) \leq k\right)^{1/q} E_\mu\left(\gamma(B_{r+1} \setminus B_r)^d\right)^{1/d}. \tag{27}$$

The proof can be completed along the lines of Proposition 3 provided we can show that $\mathcal{E}(u_r, u_r)$ is bounded in r. This will require analysis of the random variables $\gamma(\Lambda)$ on $(\ddot{\Gamma}_{\mathbb{R}^d}, \mu)$. Roughly speaking, we must show that $\gamma(\Lambda)$ is of the order $m(\Lambda)$ as $\Lambda \nearrow \mathbb{R}^d$, where m is Lebesgue measure.

For $k \in \mathbb{Z}_+$ the kth falling power of x is $x^{\underline{k}} = x(x-1)\cdots(x-(k-1))$, and the correlation function ρ_k gives

$$E_\mu\left(\gamma(\Lambda)^{\underline{k}}\right) = \int\cdots\int_{\Lambda^k} \rho_k(x_1, \ldots, x_k)\,dx_1 \ldots dx_k. \tag{28}$$

Now Ruelle [16, Proposition 2.6] shows that there exists a constant ξ so that $\rho_k \leq \xi^k$. This means that the random variable $\gamma(\Lambda)$, which has a Poisson distribution under π_{zm}, is comparable to a Poisson random variable under μ. To be precise, let's rewrite x^d in terms of falling powers of x as in

$$x^d = \sum_{k=1}^d \left\{ {d \atop k} \right\} x^{\underline{k}}, \tag{29}$$

where $\left\{ {d \atop k} \right\}$ are Stirling numbers of the second kind [5, (6.10)]. Combining (28) and (29) we get

$$\begin{aligned} E_\mu\left(\gamma(\Lambda)^d\right) &= \sum_{k=1}^d \left\{ {d \atop k} \right\} E_\mu\left(\gamma(\Lambda)^{\underline{k}}\right) \\ &\leq \sum_{k=1}^d \left\{ {d \atop k} \right\} (\xi m(\Lambda))^k \\ &= E(Z^d_{\xi m(\Lambda)}) \\ &\leq c\,E(Z^d_1)(\xi m(\Lambda))^d, \end{aligned}$$

where Z_ν denotes a Poisson random variable with mean ν, and where we must assume that $m(\Lambda)$ is bounded away from zero for the final inequality. In particular, this establishes the bound, for $m(\Lambda)$ bounded away from zero,

$$E_\mu \left(\gamma(\Lambda)^d\right)^{1/d} \leq c\,\xi\, m(\Lambda). \tag{30}$$

The most difficult part of the proof is to establish bounds for the probabilities $\mu(\gamma(\Lambda) \leq k)$, in other words, to show that Ruelle measures do not allow too few particles per unit volume in $\mathbb{R}^d$. If we assume that z is sufficiently small, then by [16, Theorems 5.7 and 5.8], μ is the unique Gibbs measure for ϕ and z, and is translation invariant. In particular, the first two correlation functions satisfy $\rho_1(x) = \rho$ for some constant, and $\rho_2(x, y) = \varphi(x - y)$ for some function $\varphi : \mathbb{R}^d \to \mathbb{R}$. First of all, (7) shows that the mean of μ is ρ times Lebesgue measure, so that $E_\mu(\gamma(\Lambda)) = \rho m(\Lambda)$. In addition, [16, Theorem 4.4.8] shows that φ is absolutely integrable on $\mathbb{R}^d$, so that using (8) we obtain

$$E_\mu(\gamma(\Lambda)^2 - \gamma(\Lambda)) = \iint_{\Lambda^2} \varphi(x - y)\, dxdy \leq cm(\Lambda).$$

This gives $\mathrm{Var}_\mu\left(\gamma(\Lambda)\right) \leq cm(\Lambda)$, and now Chebyshev's inequality implies

$$\mu(\gamma(\Lambda) \leq k) \leq \frac{c}{m(\Lambda)}. \tag{31}$$

Combining (31) and (30) with the inequality (27) shows that $(u_r)_{r\geq 0}$ is $\mathcal{E}$-bounded, and the conclusion follows. $\Box$

In the free case, under suitable growth conditions on σ, we have seen that if $\sigma(\mathbb{R}^d) = \infty$, then the number of particles in the random configuration X_t remains infinite at all times. The following proposition gives more detailed information on the growth of X_t.

LEMMA 4 (Free case) If $\sigma(\mathbb{R}^d) = \infty$, then $\lim_{r\to\infty} \gamma(B_r)/\sigma(B_r) = z$, for $\pi_{z\sigma}$-almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$.

Proof. Choose radii $(r_n)_{n\in\mathbb{N}}$ so that $\sigma(B_{r_n}) = n$. Then under the measure $\pi_{z\sigma}$, the function $\gamma(B_{r_n}) = \gamma(S_1) + \gamma(S_2 \setminus S_1) + \cdots + \gamma(B_{r_n} \setminus B_{r_n - 1})$ is the sum of n independent Poisson(z) random variables. By the law of large numbers, $\gamma(B_{r_n})/n \to z$ $\pi_{z\sigma}$-almost surely. Now for $r_n \leq r \leq r_{n+1}$ we have

$$\frac{n}{n+1}\,\frac{\gamma(B_{r_n})}{n} \leq \frac{\gamma(B_r)}{\sigma(B_r)} \leq \frac{\gamma(B_{r_{n+1}})}{n+1}\,\frac{n+1}{n} \tag{32}$$

and so the limit is attained over the continuous index r as well. $\Box$

As the following result shows, under certain conditions, the result is also true $\mathcal{E}$-quasi-everywhere; that is, the "intensity" of X_t is constant along sample paths.

PROPOSITION 5 (Free case) Suppose that $\sigma(\mathbb{R}^d) = \infty$,

$$\lim_{\epsilon \to 0} \liminf_{r \to \infty} \sigma(B_{r-\epsilon})/\sigma(B_r) = 1, \tag{33}$$

and

$$\int_{\mathbb{R}_+} z \log^+(z)\, \lambda(dz) < \infty. \tag{34}$$

Then, for μ-almost every $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$,

$$P_\gamma \left(\lim_{r\to\infty} \frac{X_t(B_r)}{\sigma(B_r)} = \lim_{r\to\infty} \frac{\gamma(B_r)}{\sigma(B_r)} \text{ for all } t \geq 0 \right) = 1.$$

Proof. For every $r > 0$ and $\epsilon > 0$ let $\psi_{r,\epsilon}$ be a smooth function satisfying $1_{(-\infty, r-\epsilon]} \leq \psi_{r,\epsilon} \leq 1_{(-\infty, r]}$, and $|\psi'_{r,\epsilon}| \leq c/\epsilon$. Define a continuous element of $D(\mathcal{E})$ by

$$u_{r,\epsilon}(\gamma) := \langle \psi_{r,\epsilon}(|\cdot|), \gamma \rangle / \sigma(B_r). \tag{35}$$

Bounding the square field gives

$$\boxplus(u_{r,\epsilon})(\gamma) \leq \frac{c^2}{\epsilon^2} \frac{\gamma(B_r)}{\sigma(B_r)^2}. \tag{36}$$

As in the previous lemma, define radii $(r_n)_{n\in\mathbb{N}}$ so that $\sigma(B_{r_n}) = n$. It is known [9] that therefore,

$$\begin{aligned} \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{n\in\mathbb{N}} \frac{\gamma(B_{r_n})}{n}\, \pi_{z\sigma}(d\gamma) &\leq c + c \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \gamma(B_{r_1}) \log^+(\gamma(B_{r_1}))\, \pi_{z\sigma}(d\gamma) \\ &\leq c + c \left(z \log^+(z) \right). \end{aligned}$$

From the inequality (32), we get

$$\int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{r \geq r_1} \frac{\gamma(B_r)}{\sigma(B_r)}\, \pi_{z\sigma}(d\gamma) \leq c + c\left(z \log^+(z)\right),$$

and hence, integrating with respect to $\lambda(dz)$ and using (34), it follows that

$$\int_{\ddot{\Gamma}_{\mathbb{R}^d}} \sup_{r \geq r_1} \frac{\gamma(B_r)}{\sigma(B_r)}\, \mu(d\gamma) < \infty.$$

Let's denote the random variable $X^*(\gamma) := \sup_{r \geq r_1} \gamma(B_r)/\sigma(B_r)$.

For fixed $n \geq r_1$, let $(A_j)_{j\in\mathbb{N}}$ be an increasing sequence of finite subsets of $[n, \infty)$ so that $\cup_j A_j$ is dense in $[n, \infty)$. For fixed γ and ϵ, the function $r \mapsto u_{r,\epsilon}(\gamma)$ is continuous on $[r_1, \infty)$ and so

$$\sup_{r \geq n} u_{r,\epsilon}(\gamma) = \sup_j \sup_{r \in A_j} u_{r,\epsilon}(\gamma).$$

Now for each $j \in \mathbb{N}$, $\sup_{r\in A_j} u_{r,\epsilon}(\gamma) \in D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. Repeated use of the inequality (12) combined with the bound (36) gives

$$\boxplus(\sup_{r\in A_j} u_{r,\epsilon}) \leq c^2 X^*/\epsilon^2\sigma(S_n),$$

and so

$$\sup_j \mathcal{E}\Big(\sup_{r\in A_j} u_{r,\epsilon}, \sup_{r\in A_j} u_{r,\epsilon}\Big) \leq \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \frac{c^2 X^*(\gamma)}{\epsilon^2\sigma(S_n)}\, \mu(d\gamma) < \infty.$$

Applying Lemma 1, we see that the pointwise limit $\sup_{r\geq n} u_{r,\epsilon}$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. In addition, the bound for the square field also carries over; $\boxplus(\sup_{r\geq n} u_{r,\epsilon}) \leq c^2X^*/\epsilon^2\sigma(S_n)$. Applying the same argument to the decreasing sequence $(\sup_{r\geq n} u_{r,\epsilon})_{n\in\mathbb{N}}$, we find that the pointwise limit $u_\epsilon := \limsup_{r\to\infty} u_{r,\epsilon}$ belongs to $D(\mathcal{E})$, is $\mathcal{E}$-quasi-continuous, and has $\boxplus(u_\epsilon) = 0$. The extra factor of $\sigma(S_n)$ in the denominator accounts for the fact that the square field is zero in the limit. Since $\mathcal{E}(u_\epsilon, u_\epsilon) = 0$ is bounded in ϵ, we may apply Lemma 1 to conclude that $u := \lim_{\epsilon\to 0} u_\epsilon$ belongs to $D(\mathcal{E})$, is $\mathcal{E}$-quasi-continuous, and has $\boxplus(u) = 0$. The regularity assumption (33) on the measure σ means that

$$u(\gamma) = \limsup_{r\to\infty} \gamma(B_r)/\sigma(B_r).$$

For any two rational numbers $0 < a < b$, we let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of smooth, compactly supported functions that vanish at the origin, decreasing pointwise to the indicator function $1_{[a,b]}$. The bound (11) shows us that $\psi_n(u)$ belongs to $D(\mathcal{E})$, is $\mathcal{E}$-quasi-continuous, and has $\boxplus(\psi_n(u)) = 0$. Letting $n \to \infty$ and applying Lemma 1 once more, we find that $1_{[a,b]}(u)$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous.

Applying the continuity result (13) simultaneously to the countable set of functions $\{1_{[a,b]}(u) : 0 < a < b \in \mathbb{Q}\}$, we conclude that the value of $\limsup_{r\to\infty} X_t(B_r)/\sigma(B_r)$ is almost surely constant in t.

A parallel argument shows that $\liminf_{r\to\infty} X_t(B_r)/\sigma(B_r)$ is also almost surely constant in t, and this gives the result. □

COROLLARY 2 The condition $\int_{\mathbb{R}_+} z^2\, \lambda(dz) < \infty$ in Proposition 1 can be dropped if $\limsup_r \sigma(B_{r-\epsilon})/\sigma(B_r) > 0$ for some $\epsilon > 0$.

Proof. Since the case $\sigma(\mathbb{R}^d) < \infty$ was covered in Corollary 1, we will assume that $\sigma(\mathbb{R}^d) = \infty$. Choose $\epsilon > 0$ and $(r_k)_{k\in\mathbb{N}}$ so that $r_k \uparrow \infty$, $\sum_k \sigma(B_{r_k})^{-1} < \infty$, and $\lim_{k\to\infty} \sigma(B_{r_k-\epsilon})/\sigma(B_{r_k}) = c > 0$. Define v_k to be the function $u_{r_k,\epsilon}$ as defined in (35). We have

$$\boxplus\Big(\sup_k v_k\Big) \leq \sup_k \boxplus(v_k) \leq \sum_k \boxplus(v_k),$$

and using the bound (36) and the condition $\sum_k \sigma(B_{r_k})^{-1} < \infty$, we find that the right-hand side is μ-integrable. As in the proof of Proposition 5, we conclude that $\limsup_k v_k$ belongs to $D(\mathcal{E})$ and $\boxplus(\limsup_k v_k) = 0$, so that for any $b > 0$, $v := 1_{[0,b]}(\limsup_k v_k)$ belongs to $D(\mathcal{E})$ and $\boxplus(v) = 0$.

Now apply the product formula (25) to $u_n v$ where the functions $(u_n)_{n \in \mathbb{N}}$ are defined in (16). Since

$$\limsup_k v_k(\gamma) \geq c \limsup_k \gamma(B_{r_k - \epsilon})/\sigma(B_{r_k - \epsilon}),$$

we know from Lemma 4 that v vanishes $\pi_{z\sigma}$ almost every for $z > b/c$. In particular,

$$\begin{aligned} \mathcal{E}(u_n v, u_n v) &= \int_{\mathbb{R}_+} \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \boxplus(u_n v)(\gamma)\, \pi_{z\sigma}(d\gamma)\lambda(dz) \\ &= \int_{\mathbb{R}_+} \int_{\{\liminf_k v_k \leq b\}} \boxplus(u_n)(\gamma)\, \pi_{z\sigma}(d\gamma)\lambda(dz) \\ &\leq \int_0^{b/c} \int_{\ddot{\Gamma}_{\mathbb{R}^d}} \boxplus(u_n)(\gamma)\, \pi_{z\sigma}(d\gamma)\lambda(dz). \end{aligned}$$

So as in the proof of Proposition 1, with the measure λ effectively truncated at the value b/c, we see that $(u_n\, v)_{n \in \mathbb{N}}$ is $\mathcal{E}$-bounded and so conclude that the set $N \cap \{\limsup_k v_k \leq b\}$ is $\mathcal{E}$-exceptional. Since b is arbitrary, this yields the desired result. □

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