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Generalized Mehler Semigroups and Catalytic Branching Processes with Immigration

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Abstract

A generalized Mehler semigroup (Ornstein-Uhlenbeck semigroup) associated with some strongly continuous semigroup of linear operators on a real separable Hilbert space may be defined by using a skew convolution semigroup. Under a mild moment assumption, it is proved that the characteristic functional of any centered skew convolution semigroup is absolutely continuous and characterizations are given for centered skew convolution semigroups whose characteristic functionals are not necessarily differentiable at the initial time. A connection between the subject and measure-valued catalytic branching processes is established by using fluctuation limits of their associated immigration processes. Path regularity of the generalized the corresponding Ornstein-Uhlenbeck processes in different topologies is also discussed.

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branching process, absorbing barrier Brownian motion, catalytic branching, fluctuation limit.

1 Introduction

Suppose that E is a real separable Hilbert space with dual space E^*. We use $\hat{m}(a)$, $a \in E^*$, to denote the characteristic functional of a given Borel probability measure m on E. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of linear operators on E with dual $(T_t^*)_{t\geq 0}$ and let $(\mu_t)_{t\geq 0}$ be a family of probability measures on E. The family $(\mu_t)_{t\geq 0}$ is called a *skew convolution semigroup* associated with $(T_t)_{t\geq 0}$ if the following equation is satisfied:

$$\mu_{r+t} = (T_t \mu_r) * \mu_t, \qquad r, t \geq 0, \tag{1.1}$$

where "$*$" denotes the convolution operation. The above equation holds if and only if we can define a Markov transition semigroup $(P_t)_{t\geq 0}$ on E by

$$P_t f(x) := \int_E f(T_t x + y)\mu_t(dy), \quad x \in E, f \in B(E), \tag{1.2}$$

where $B(E)$ denotes the totality of bounded Borel measurable functions on E. In this case, $(P_t)_{t\geq 0}$ is called a *generalized Mehler semigroup* or *generalized Ornstein-Uhlenbeck semigroup*, which corresponds to an Ornstein-Uhlenbeck process with state space E. The above formulation of Ornstein-Uhlenbeck processes was given by Bogachev et al (1996). One motivation to study those processes is that they constitute a large class of explicit examples of processes on infinite-dimensional spaces. We refer the reader to Bogachev et al (1996) and Fuhrman and Röckner (2000) for the discussion from theoretical viewpoints. In particular, it was proved in Bogachev et al (1996, Lemma 2.6 and Proposition 4.3) that if, for all $a \in E^*$, the function $t \mapsto \hat{\mu}_t(a)$ is absolutely continuous on $[0, \infty)$ and differentiable at $t = 0$ and if, setting $\lambda(a) := -(d/dt)\hat{\mu}_t(a)|_{t=0}$, the function $t \mapsto \lambda(T_t^* a)$ is locally Lebesgue integrable on $[0, \infty)$, then equality (1.1) is equivalent to

$$\hat{\mu}_t(a) = \exp\Big\{ - \int_0^t \lambda(T_s^* a) ds \Big\}, \quad t \geq 0, a \in E^*, \tag{1.3}$$

and in this case λ is negative definite. This result gives a characterization for a special class of generalized Mehler semigroups defined by (1.2).

There is also a body of work that looks at a similar equation in measure-valued setting, motivated by the study of immigration structures associated with branching systems. Suppose that F is a Lusin topological space with Borel σ-algebra $\mathcal{B}(F)$. We denote by $B(F)^+$ the set of bounded non-negative $\mathcal{B}(F)$-measurable functions on F.

Let $M(F)$ be the totality of finite measures on $(E, \mathcal{B}(F))$ endowed with the topology of weak convergence. Set $\mu(f) = \int_E f d\mu$ for $f \in B(F)^+$ and $\mu \in M(F)$. A Borel right Markov process X with state space $M(F)$ and transition semigroup $(Q_t)_{t\geq 0}$ is called a regular *measure-valued branching process (superprocess)* if there is a locally bounded semigroup of operators $(V_t)_{t\geq 0}$ on $B(F)^+$ such that

$$\int_{M(F)} e^{-\nu(f)} Q_t(\mu, d\nu) = \exp\left\{-\mu(V_t f)\right\}, \qquad t \geq 0, \mu \in M(F), f \in B(F)^+. \tag{1.4}$$

In this case, $(V_t)_{t\geq 0}$ is called the *cumulant semigroup* of X. A class of measure-valued immigration processes was formulated in Li (1995/1996, 1996) as follows. A family $(N_t)_{t\geq 0}$ of probability measures on $M(F)$ is called a *skew convolution semigroup* associated with $(Q_t)_{t\geq 0}$ if it satisfies

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r, t \geq 0. \tag{1.5}$$

We use the same terminology for solutions of (1.1) and those of (1.5) since formally (1.1) is a special case of (1.5). The similarity between the two equations was first noticed by L.G. Gorostiza. Equation (1.5) holds if and only if

$$Q_t^N(\mu, \cdot) := Q_t(\mu, \cdot) * N_t, \qquad t \geq 0, \mu \in M(F), \tag{1.6}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(F)$. If Y is a Markov process in $M(F)$ having transition semigroup $(Q_t^N)_{t\geq 0}$, we call it an *immigration process* associated with X. It was proved in Li (1995/1996) that the family of probability measures $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$ if and only if there is an infinitely divisible probability entrance law $(K_s)_{s>0}$ for $(Q_t)_{t\geq 0}$ such that

$$\log \int_{M(F)} e^{-\nu(f)} N_t(d\nu) = \int_0^t \Big[\log \int_{M(F)} e^{-\nu(f)} K_s(d\nu)\Big] ds, \quad t \geq 0, f \in B(F)^+. \tag{1.7}$$

This result gives a complete characterization for the solutions of (1.5). Some representations of the infinitely divisible probability entrance laws and path regularity of the corresponding immigration processes were studied in Li (1996, 1998b). The corresponding theory for branching particle systems was developed in Li (1998a) and a time inhomogeneous version of (1.5) was discussed in Li (2001). The connection between measure-valued immigration processes and generalized Ornstein-Uhlenbeck processes was studied in Gorostiza and Li (1998, 1999) and Li (1999).

Recently, Schmuland and Sun (2001) proved that the probability measures $(\mu_t)_{t\geq 0}$ satisfying (1.1) are always infinitely divisible and, in some typical cases, the function $t \mapsto \hat{\mu}_t(a)$ is continuous on $[0, \infty)$. By (1.7), the function

$$t \mapsto \int_{M(F)} e^{-\nu(f)} N_t(d\nu) \tag{1.8}$$

is differentiable at $t = 0$ for all continuous $f \in B(F)^+$ if and nearly only if $(K_s)_{s>0}$ can be closed by an infinitely divisible probability measure K_0 on $M(F)$. Therefore, one naturally expects that equation (1.1) could have solutions for which $t \mapsto \hat{\mu}_t(a)$ is not differentiable at $t = 0$ and, under suitable regularity conditions, the absolute continuity of $t \mapsto \hat{\mu}_t(a)$ would follow automatically from (1.1). This observation provided one motivation for the present work.

Another motivation of the work came from the study of a class of catalytic branching superprocesses described as follows. Let $D \subset \mathbb{R}^d$ be a domain with smooth boundary. Suppose that L is a uniformly elliptic differential operator on D with Dirichlet boundary conditions. Let $\eta \in B(D)^+$ and let $\phi(\cdot,\cdot)$ be a function on $D \times [0,\infty)$ of a certain form to be specified. It is known that

$$\begin{aligned} \frac{\partial}{\partial t} V_t f(x) &= L V_t f(x) - \eta(x)\phi(x, V_t f(x)), \\ V_0 f(x) &= f(x), \end{aligned} \tag{1.9}$$

defines the cumulant semigroup $(V_t)_{t\geq 0}$ of a superprocess with state space $M(D)$; see e.g. Dawson (1993). The process describes the catalytic reaction of a large number of infinitesimal particles moving according to the transition law of the diffusion process generated by L and splitting according to the branching mechanism given by $\phi(\cdot,\cdot)$. The value $\eta(x)$ represents the density at $x \in D$ of a catalyst which causes the splitting. However, there are some catalytic reactions in which the catalyst is concentrated on a very small set and in that case the coefficient $\eta(\cdot)$ has to be replaced by an irregular one; see e.g. Pagliaro and Taylor (1988). These lead to the study of a catalyst given not by a regular density function but rather by a measure $\eta(dx) :=$ catalytic mass in the volume element dx. Then we may reformulate (1.9) into the corresponding mild form

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_D \phi(y, V_s f(y)) p_{t-s}(x,y)\eta(dy), \quad t \geq 0, x \in D, \tag{1.10}$$

where $(P_t)_{t\geq 0}$ and $(p_t)_{t>0}$ denote respectively the transition semigroup and the transition density of the diffusion process generated by L. The study of branching models with irregular catalysts was initiated by Dawson and Fleischmann (1991, 1992) and there has been a considerable development and literature in the mathematical theory since then; see e.g. Dawson and Fleischmann (1999) for a recent survey. We shall consider in detail the situation where $D = (0,\infty)$.

In this paper, we give some basic characterizations for the skew convolution semigroups defined by (1.1) and the corresponding generalized Ornstein-Uhlenbeck processes. Since a non-centered solution of (1.1) may behave very irregularly as observed in Schmuland and Sun (2001), we shall only consider centered solutions. Under a mild moment assumption, we show that the absolute continuity of $t \mapsto \hat{\mu}_t(a)$ does follow from (1.1) and we give characterizations for solutions $(\mu_t)_{t\geq 0}$ for which the functions $t \mapsto \hat{\mu}_t(a)$ are not necessarily differentiable at $t = 0$. An important feature of the Ornstein-Uhlenbeck processes corresponding to such solutions is that they have right continuous strong Markov realizations only when the state space E is suitably extended, which is quite similar to the situation of the immigration processes investigated in Li

(1996, 1998a, 2001). We discuss the regularity of the Ornstein-Uhlenbeck processes in the particular case where $E = L^2(0,\infty)$ and $(T_t)_{t\geq 0}$ is the transition semigroup of the absorbing barrier Brownian motion (ABM) in $(0,\infty)$. We shall see that the corresponding Ornstein-Uhlenbeck processes arise naturally in the study of fluctuation limits of immigration processes associated with catalytic branching super ABM over $(0,\infty)$. The Ornstein-Uhlenbeck processes with state space $L^2(0,\infty)$ usually do not have right continuous sample paths, neither do they have the strong Markov property. Nevertheless, we show that some of them do have the regularity if we regard them as processes with values of signed-measures. The study of generalized Mehler semigroups on Hilbert spaces and that of catalytic branching processes have been going independently of each other with different motivations, techniques, and so on. Our results establish some connection between the two subjects and give interpretations of the abstract results on generalized Mehler semigroups from the viewpoint of applications. In the connection, important roles are played by the measure-valued immigration processes formulated in Li (1995/1996, 1996) and their small branching fluctuation limits of the type of Gorostiza (1996), Gorostiza and Li (1998) and Li (1999, 2000).

2 Characterization of skew convolution semigroups

By a result of Schmuland and Sun (2001), if $(\mu_t)_{t\geq 0}$ is a solution of (1.1), then each μ_t is an infinitely divisible probability measure. Let

$$K_1(a,x) := e^{i\langle x,a\rangle} - 1 - i\langle x,a\rangle, \qquad x\in E, a\in E^*.$$

We shall only consider infinitely divisible probability measures $(\mu_t)_{t\geq 0}$ with characteristic functionals represented by

$$\hat{\mu}_t(a) = \exp\Big\{ -\frac{1}{2}\langle R_t a, a\rangle + \int_{E^\circ} K_1(a,x) M_t(dx)\Big\}, \qquad t\geq 0, a\in E^*, \tag{2.1}$$

where $\{R_t : t\geq 0\}$ is a family of nuclear operators on E and $t\mapsto (\|x\|\wedge\|x\|^2)M_t(dx)$ is a finite kernel from $[0,\infty)$ to $E^\circ := E\setminus\{0\}$. (This integral condition is slightly stronger than the one usually required for a Lévy measure; see e.g. Linde (1986, page 75).) In particular, $(\mu_t)_{t\geq 0}$ are centered probability measures. Let $(\mu_t^g)_{t\geq 0}$ and $(\mu_t^p)_{t\geq 0}$ denote respectively the centered Gaussian and the centered Poisson type factors of $(\mu_t)_{t\geq 0}$. It is easy to check that $(\mu_t)_{t\geq 0}$ satisfies (1.1) if and only if both $(\mu_t^g)_{t\geq 0}$ and $(\mu_t^p)_{t\geq 0}$ satisfy the same equation. Indeed, $(\mu_t^g)_{t\geq 0}$ satisfies (1.1) if and only if

$$R_{r+t} = T_t R_r T_t^* + R_t, \qquad r,t\geq 0 \tag{2.2}$$

and $(\mu_t^p)_{t\geq 0}$ satisfies (1.1) if and only if

$$M_{r+t} = (T_t M_r)|_{E^\circ} + M_t, \qquad r,t\geq 0. \tag{2.3}$$

Theorem 2.1 *A family of probability measures $(\mu_t)_{t\geq 0}$ on E is a centered Gaussian type solution of (1.1) if and only if its characteristic functionals are represented by*

$$\hat{\mu}_t(a) = \exp\Big\{ -\frac{1}{2}\int_0^t \langle U_s a, a\rangle ds\Big\}, \qquad t\geq 0, a\in E^*, \tag{2.4}$$

where $\{U_s : s>0\}$ is a family of nuclear operators on E satisfying $U_{s+t} = T_t U_s T_t^*$ for all $s,t>0$ and

$$\int_0^t \mathrm{Tr} U_s ds < \infty, \qquad t\geq 0.$$

Proof. Suppose $(\mu_t)_{t\geq 0}$ is given by (2.4) and let $R_t = \int_0^t U_s ds$. Under the assumptions, $(R_t)_{t\geq 0}$ satisfies (2.2), so $(\mu_t)_{t\geq 0}$ is a centered Gaussian type solution of (1.1). To prove the inverse assertion, suppose $(\mu_t)_{t\geq 0}$ is a centered Gaussian type solution of (1.1). Then we have

$$\int_E \|x\|^2 \mu_{r+t}(dx) = \int_E \|T_t x\|^2 \mu_r(dx) + \int_E \|x\|^2\mu_t(dx), \qquad r,t\geq 0. \tag{2.5}$$

It follows that

$$g(t) := \int_E \|x\|^2 \mu_t(dx), \qquad t\geq 0, \tag{2.6}$$

is a non-decreasing function. Since $(T_t)_{t\geq 0}$ is strongly continuous, there are constants $c\geq 1$ and $b\geq 0$ such that $\|T_t\|\leq ce^{bt}$. We claim that, for $0<r_1<t_1<\cdots<r_n<t_n\leq l$,

$$\sum_{j=1}^n [g(t_j) - g(r_j)] \leq c^2 e^{2bl} g(\sigma_n), \tag{2.7}$$

where $\sigma_n = \sum_{j=1}^n (t_j - r_j)$. When $n=1$, this follows from (2.5). Now assume (2.7) holds for $n-1$. Applying (2.5) twice,

$$\begin{aligned}
\sum_{j=1}^n [g(t_j) - g(r_j)] &\leq [g(t_n) - g(r_n)] + c^2 e^{2bl}\int_E \|x\|^2 \mu_{\sigma_{n-1}}(dx) \\
&= \int_E \|T_{r_n} x\|^2 \mu_{t_n - r_n}(dx) + c^2 e^{2bl}\int_E \|x\|^2 \mu_{\sigma_{n-1}}(dx) \\
&\leq c^2 e^{2bl}\int_E \|T_{\sigma_{n-1}} x\|^2 \mu_{t_n-r_n}(dx) + c^2 e^{2bl}\int_E \|x\|^2\mu_{\sigma_{n-1}}(dx) \\
&= c^2 e^{2bl}\int_E \|x\|^2 \mu_{\sigma_n}(dx),
\end{aligned}$$

giving (2.7). Letting $r\to 0$ and $t\to 0$ in (2.5) and using the fact that $g(\cdot)$ is a non-decreasing function one sees $g(t)\to 0$ as $t\to 0$. By this and (2.7), $g(t)$ is absolutely continuous in $t \geq 0$. From (2.2) we see that $\langle R_t a, a\rangle$ is a non-decreasing function of $t \geq 0$ for any $a \in E^*$. For $t \geq r \geq 0$, (2.2) yields

$$\begin{aligned}\langle R_t a, a\rangle - \langle R_r a, a\rangle &= \langle R_{t-r} T_r^* a, T_r^* a\rangle = \int_E \langle x, T_r^* a\rangle^2 \mu_{t-r}(dx) \\ &\leq \|a\|^2 \int_E \|T_r x\|^2 \mu_{t-r}(dx) = \|a\|^2 [g(t) - g(r)].\end{aligned}$$

Then $\langle R_t a, a\rangle$ is absolutely continuous in $t \geq 0$. Now it is simple to see that $\langle R_t a, b\rangle$ is absolutely continuous in $t \geq 0$ for all a and $b \in E^*$. Let $\{e_n : n = 1, 2, \ldots\}$ be an orthonormal basis of $E = E^*$. From the above arguments, there are locally integrable functions $A_{m,n}(\cdot)$ on $[0, \infty)$ such that

$$\langle R_t e_m, e_n\rangle = \int_0^t A_{m,n}(s) ds, \qquad t \geq 0,\ m, n \geq 1. \tag{2.8}$$

¿From the symmetry of R_t we get

$$\int_0^t A_{m,n}(s) ds = \int_0^t A_{n,m}(s) ds, \tag{2.9}$$

while the positivity of R_t gives

$$\langle R_t a, a\rangle = \int_0^t \sum_{m,n=1}^\infty A_{m,n}(s) \langle a, e_m\rangle \langle a, e_n\rangle ds \geq 0 \tag{2.10}$$

for $a \in \operatorname{span}\{e_1, e_2, \ldots\}$. (The sum is actually finite!) In addition, since R_t is nuclear we have

$$\operatorname{Tr}(R_t) = \sum_{n=1}^\infty \langle R_t e_n, e_n\rangle = \int_0^\infty \Big(\sum_{n=1}^\infty A_{n,n}(s)\Big) ds < \infty. \tag{2.11}$$

Let F be the Borel subset of $[0, \infty)$ consisting of $s \geq 0$ such that $A_{m,n}(s) = A_{n,m}(s)$ for $m, n \geq 1$ and

$$\sum_{m,n=1}^\infty A_{m,n}(s) \langle a, e_m\rangle \langle a, e_n\rangle \geq 0 \quad \text{and} \quad \sum_{n=1}^\infty A_{n,n}(s) < \infty \tag{2.12}$$

for $a \in \operatorname{span}\{e_1, e_2, \ldots\}$ with rational coefficients. By (2.9), (2.10) and (2.11), F has full Lebesgue measure. For any $s \in F$,

$$U_s a = \sum_{m,n=1}^\infty A_{m,n}(s) \langle a, e_m\rangle e_n, \qquad a \in E, \tag{2.13}$$

defines a positive, symmetric linear operator on $\mathrm{span}\{e_1, e_2, \ldots\}$ with $\langle U_s e_m, e_n\rangle = A_{m,n}(s)$. Taking $b = xe_m + ye_n$ we get

$$\langle U_s b, b\rangle = (x \;\; y)\begin{pmatrix} A_{m,m}(s) & A_{m,n}(s) \\ A_{n,m}(s) & A_{n,n}(s) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \geq 0,$$

so that the 2×2 matrix above is non-negative definite. Therefore its determinant is non-negative, that is,

$$A_{m,n}(s)^2 \leq A_{m,m}(s)A_{n,n}(s). \tag{2.14}$$

By this and Schwarz inequality,

$$\begin{aligned} \|U_s a\|^2 &= \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} A_{m,n}(s)\langle a, e_m\rangle\Big)^2 \\ &\leq \sum_{n=1}^{\infty} A_{n,n}(s)\Big(\sum_{m=1}^{\infty} A_{m,m}(s)^{1/2}|\langle a, e_m\rangle|\Big)^2 \\ &\leq \Big(\sum_{n=1}^{\infty} A_{n,n}(s)\Big)^2\|a\|^2 \end{aligned}$$

for $s \in F$. This shows that U_s is a bounded operator and can be extended to the entire space E. In fact, U_s is a nuclear operator since

$$\mathrm{Tr}(U_s) = \sum_{n=1}^{\infty}\langle U_s e_n, e_n\rangle = \sum_{n=1}^{\infty} A_{n,n}(s) < \infty.$$

By (2.8) and (2.13), for $a \in \mathrm{span}\{e_1, e_2, \ldots\}$ we have

$$\langle R_t a, a\rangle = \sum_{m,n=1}^{\infty}\langle a, e_m\rangle\langle a, e_n\rangle\langle R_t e_m, e_n\rangle = \int_0^t \langle U_s a, a\rangle ds, \qquad t \geq 0. \tag{2.15}$$

Since $s \mapsto \mathrm{Tr}(U_s)$ is locally integrable, by dominated convergence we see that (2.15) holds for all $a \in E^*$. Now (2.2) implies that

$$\int_0^r \langle U_{s+t}a, a\rangle ds = \int_0^r \langle U_s T_t^* a, T_t^* a\rangle ds, \qquad r, t \geq 0, a \in E^*.$$

Since E^* is separable, by Fubini's theorem, there are subsets H and H_s of $[0, \infty)$ with full Lebesgue measure such that

$$U_{s+t} = T_t U_s T_t^*, \qquad s \in H, t \in H_s.$$

Choose a decreasing sequence $s_n \in H$ with $s_n \to 0$, and define

$$U_t = T_{t-s_n} U_{s_n} T_{t-s_n}^*, \qquad t > s_n.$$

Under this modification, $(U_t)_{t>0}$ satisfies $U_{r+t} = T_t U_s T_t^*$ for all $r, t > 0$, while (2.15) remains unchanged. □

Theorem 2.2 *A family of probability measures $(\mu_t)_{t\geq 0}$ on E is a centered Poisson type solution of (1.1) if and only if its characteristic functionals are represented by*

$$\hat{\mu}_t(a) = \exp\Big\{\int_0^t ds \int_{E^\circ} K_1(a,x)L_s(dx)\Big\}, \quad t \geq 0, a \in E^*, \tag{2.16}$$

where $L_s(dx)$ is a σ-finite kernel from $(0,\infty)$ to E° satisfying $L_{r+t} = (T_t L_r)|_{E^\circ}$ for all $r, t > 0$ and

$$\int_0^t ds \int_E (\|x\| \wedge \|x\|^2)L_s(dx) < \infty, \qquad t \geq 0.$$

Proof. It is easy to see that if $(\mu_t)_{t\geq 0}$ is given by (2.16), then it is a centered Poisson type solution of (1.1). Now suppose $(\mu_t)_{t\geq 0}$ is a centered Poisson type solution of (1.1) with Lévy measures $(M_t)_{t\geq 0}$. We shall prove $(\hat{\mu}_t)_{t\geq 0}$ is of the form (2.16). From (2.3) we see that M_t is non-decreasing in $t \geq 0$. Let $c \geq 1$ and $b \geq 0$ be as in the proof of Theorem 2.1 and let

$$h(t) := \int_{E^\circ} (\|x\| \wedge \|x\|^2)M_t(dx), \qquad t \geq 0.$$

By (2.3) we have, for $t, r \geq 0$,

$$h(r+t) - h(r) = \int_{E^\circ} (\|T_r x\| \wedge \|T_r x\|^2)M_t(dx),$$

which is bounded above by $c^2 e^{2br} g(r)$. As in the proof of Theorem 2.1, one sees that $h(t)$ is absolutely continuous in $t \geq 0$. Since the family of finite measure $\nu_t(dx) := (\|x\| \wedge \|x\|^2)M_t(dx)$ is non-decreasing and since $t \mapsto h(t) = \nu_t(E^\circ)$ is absolutely continuous, $\nu([0,t], B) = \nu_t(B)$ defines a locally bounded Borel measure $\nu(\cdot, B)$ on $[0,\infty)$ for each $B \in \mathcal{B}(E^\circ)$. A monotone class argument shows that $\nu(A, \cdot)$ is a Borel measure on E° for each $A \in \mathcal{B}([0,\infty))$, so that $\nu(\cdot,\cdot)$ is a bimeasure. By Ethier and Kurtz (1996, page 502), there is a probability kernel $J_s(dx)$ from $[0,\infty)$ to E° such that

$$\nu(A,B) = \int_A J_s(B)\nu(ds, E^\circ) = \int_A J_s(B)dh(s) = \int_A J_s(B)h'(s)ds,$$

where $h'(s)$ is a Radon-Nikodym derivative of $dh(s)$ relative to the Lebesgue measure. Using the σ-finite kernel $L_s(dx) := (\|x\| \wedge \|x\|^2)^{-1}h'(s)J_s(dx)$ we obtain

$$\int_{E^\circ} K_1(a,x)M_t(dx) = \int_0^t ds \int_{E^\circ} K_1(a,x)L_s(dx) \tag{2.17}$$

for $t \geq 0$ and $a \in E^*$. Now (2.3) implies that

$$\int_0^r ds \int_{E^\circ} K_1(a,x)L_{s+t}(dx) = \int_0^r ds \int_{E^\circ} K_1(a,x)L_s(dx)$$

for all $r, t \geq 0$ and $a \in E^*$. Since E^* is separable, one can modify the definition of the family $(L_t)_{t>0}$ similarly as in the proof of Theorem 2.1 so that it satisfies $L_{r+t} = (T_t L_r)|_{E^\circ}$ for all $r, t > 0$, while (2.17) remains unchanged. □

Observe that, for the families $\{U_s : s > 0\}$ and $\{L_s : s > 0\}$ given as in Theorems 2.1 and 2.2,

$$\hat{\nu}_s(a) = \exp\Big\{ -\frac{1}{2}\langle U_s a, a\rangle + \int_{E^\circ} K_1(a, x) L_s(dx)\Big\}, \qquad s > 0, a \in E^*, \tag{2.18}$$

defines a family of centered infinitely divisible probability measures $\{\nu_s : s > 0\}$ on E satisfying $\nu_{r+t} = T_t \nu_r$ for all $r, t > 0$. Combining together Theorems 2.1 and 2.2 we get the following theorem which improves the characterizations of Bogachev et al (1996, Lemma 2.6 and Proposition 4.3) and presents a Hilbert space version of the results of Li (1995/1996, Theorem 2) and Li (2001, Theorems 3.1 and 3.7).

Theorem 2.3 *The family of centered probabilities $(\mu_t)_{t\geq 0}$ given by (2.1) is a solution of (1.1) if and only if there is a family $(\nu_s)_{s>0}$ given by (2.18) such that the characteristic functionals of $(\mu_t)_{t\geq 0}$ are represented by*

$$\hat{\mu}_t(a) = \exp\Big\{ \int_0^t \log \hat{\nu}_s(a) ds\Big\}, \qquad t \geq 0, a \in E^*, \tag{2.19}$$

where $\log \hat{\nu}_s(\cdot)$ denotes the unique continuous function on E^* with $\log \hat{\nu}_s(0) = 0$ and $\hat{\nu}_s(a) = \exp\{\log \hat{\nu}_s(a)\}$ for all $a \in E^*$.

Now let us see some particular skew convolution semigroups included in (2.19) whose characteristic functionals are not necessarily differentiable at the initial time. Let $D := (0, \infty)$ and let λ denote the one-dimensional Lebesgue measure. We shall consider the Hilbert space $L^2(D, \lambda)$. Let

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\{-x^2/2t\}, \qquad t > 0, x \in \mathbb{R},$$

and let

$$p_t(x, y) = g_t(x - y) - g_t(x + y), \qquad t > 0, x, y \in D. \tag{2.20}$$

We may define a semigroup of linear operators $(P_t)_{t\geq 0}$ on $L^2(D, \lambda)$ by $P_0 h = h$ and

$$P_t h(x) = \int_D p_t(x, y) h(y) dy, \qquad t > 0, x \in D. \tag{2.21}$$

Indeed, $(P_t)_{t\geq 0}$ is the transition semigroup of the ABM in D. Let

$$k_t(y) = 2^{-1}(d/dx) p_t(x, y)|_{x=0^+} = y g_t(y)/t, \qquad t > 0, x \in D. \tag{2.22}$$

Then we have $\int_0^\infty k_t(y)dt = 1$ and

$$k_{r+t}(y) = \int_D p_t(x,y)k_r(x)dx, \qquad r,t>0, y\in D. \tag{2.23}$$

Example 2.1 Let η be a finite measure on D and let $\varphi(\cdot,\cdot)$ be a function on $D\times I\!R$ with the representation

$$\varphi(x,z) = -c(x)z^2 + \int_{I\!R^\circ} (\mathrm{e}^{izu} - 1 - izu)m(x,du), \quad x\in D, z\in I\!R,$$

where $c \in B(D)^+$ and $u^2m(x,du)$ is a bounded kernel from D to $I\!R^\circ := I\!R\setminus\{0\}$. By Theorem 5.1, there is a skew convolution semigroup $(\mu_t)_{t\geq 0}$ on $L^2(D,\lambda)$ given by

$$\hat\mu_t(f) = \exp\Big\{\int_0^t \eta(\varphi(P_sf))ds\Big\}, \qquad t\geq 0, f\in L^2(D,\lambda). \tag{2.24}$$

Clearly, this skew convolution semigroup is a special form of the one given by (2.19). In general, $(\mu_t)_{t\geq 0}$ does not satisfy the regularity condition imposed in Bogachev et al (1996) and Fuhrman and Röckner (2000). As an example, take any $x_0\in D$ and let $\eta = \delta_{x_0}$, $\varphi(x,z)\equiv -z^2$ and $f(x) = |x^2 - x_0^2|^{-1/3}$. In this case, $\eta(\varphi(P_sf)) = -P_sf(x_0)^2 \to -\infty$ as $s\to 0^+$ and hence the function $t\mapsto \hat\mu_t(f)$ is not differentiable at $t=0$. We shall see that Ornstein-Uhlenbeck processes corresponding to skew convolution semigroups of this type arise naturally as the fluctuation limits of some immigration processes associated with the catalytic branching super ABM.

Example 2.2 Let $\varphi(\cdot)$ be given by

$$\varphi(z) = \int_{I\!R^\circ} \left(\mathrm{e}^{iuz} - 1 - iuz\right) m(du), \quad x\in D, z\in I\!R,$$

where $(1\vee|u|)m(du)$ is a finite measure on $I\!R^\circ$. By Theorem 5.3,

$$\hat\mu_t'(f) = \exp\Big\{\int_0^t \varphi(\langle k_s, f\rangle)ds\Big\}, \qquad t\geq 0, f\in L^2(D,\lambda), \tag{2.25}$$

defines a skew convolution semigroup $(\mu_t')_{t\geq 0}$ on $L^2(D,\lambda)$. This skew convolution semigroup is obtained from the last example by replacing $\varphi(x,z)$ and $\eta(dx)$ in (2.24) by $\varphi(nz)$ and $\delta_{1/2n}(dx)$, respectively, and letting $n\to\infty$. Using (2.23) one may check that $(\mu_t')_{t\geq 0}$ is included in (2.16). Unless $\varphi(\cdot)$ is trivial, $\varphi(\langle k_s, f\rangle)$ is not convergent for general $f\in L^2(D,\lambda)$ as $s\to 0^+$, so this skew convolution semigroup is not included in (1.3).

3 Catalytic branching processes with immigration

In this section, we define a catalytic branching superprocess and its immigration processes. For simplicity, we only consider the case where the underlying motion is an

ABM in $(0,\infty)$. Let $M(D)$ denote the space of finite Borel measures on $D := (0,\infty)$. We endow $M(D)$ with the topology of weak convergence. Let ϕ be given by

$$\phi(x,z) = c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)m(x,du), \quad z \geq 0, x \in D, \tag{3.1}$$

where $c \in B(D)^+$ and $u^2 m(x,du)$ is a bounded kernel from D to $(0,\infty)$. Let η be a measure on D to which there correspond constants $c = c(\eta) > 0$ and $l = l(\eta) > 0$ such that

$$\eta([x, x+l]) \leq cl, \qquad x \in D. \tag{3.2}$$

Clearly, the above condition is satisfied if η is finite or absolutely continuous with respect to the Lebesgue measure with bounded density function. Let $\{l_s(x) : x > 0, s > 0\}$ be a continuous version of the local time of an ABM B with transition semigroup $(P_t)_{t\geq 0}$ defined by (2.20) and (2.21). Then $k(t) := \eta(l_t)$ defines a continuous additive functional of B.

Lemma 3.1 *For any $x \in D$ and $t > 0$,*

$$\boldsymbol{E}_x k(t) = \int_0^t ds \int_D p_s(x,y)\eta(dy) \leq 2c\sqrt{t}\left(\sqrt{2}l + \sqrt{\pi t}\right).$$

Proof. By the definition of $k(t)$ and the assumption (3.2) we have

$$\begin{aligned}
\boldsymbol{E}_x k(t) &= \int_0^t ds \int_D p_s(x,y)\eta(dy) \\
&\leq \int_0^t ds \int_D g_s(y-x)\eta(dy) \\
&\leq \int_0^t \frac{2cl}{\sqrt{2\pi s}} \sum_{k=0}^\infty \exp\Big\{-\frac{k^2 l^2}{2s}\Big\} ds \\
&\leq \int_0^t \frac{c}{\sqrt{2\pi s}} \Big(2l + \int_{I\!R} \exp\Big\{-\frac{y^2}{2s}\Big\} dy\Big) ds \\
&\leq \int_0^t \frac{c}{\sqrt{2\pi s}} \left(2l + \sqrt{2\pi s}\right) ds \\
&\leq 2c\sqrt{t}\left(\sqrt{2}l + \sqrt{\pi t}\right),
\end{aligned}$$

as desired. □

By Lemma 3.1 and the general construction of measure-valued branching processes, there is a Markov transition semigroup $(Q_t)_{t\geq 0}$ on $M(D)$ that is determined by

$$\int_{M(D)} \mathrm{e}^{-\nu(f)} Q_t(\mu, d\nu) = \exp\{-\mu(V_t f)\}, \quad f \in B(D)^+, \tag{3.3}$$

where $V_t f$ denotes the unique positive solution of the evolution equation

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_D \phi(y, V_s f(y)) p_{t-s}(x,y)\eta(dy), \quad t \geq 0, x \in D; \tag{3.4}$$

see e.g. Dawson (1993), Dynkin (1994) or Leduc (2000). A Markov process X on $M(D)$ with transition semigroup $(Q_t)_{t\geq 0}$ is called a *catalytic branching super ABM* with parameters (η, ϕ). The catalytic branching process was introduced by Dawson and Fleischmann (1991, 1992) as a stochastic model for physiochemical and biological systems and it has been studied by many others; see Dawson and Fleischmann (1999) for a recent survey.

Let $\kappa_t(dx) = k_t(x)dx$. By (2.23), $(\kappa_t)_{t>0}$ form an entrance law for the underlying semigroup $(P_t)_{t\geq 0}$, that is, $\kappa_r P_t = \kappa_{r+t}$ for all $r, t > 0$. Let

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t ds \int_D \phi(y, V_s f(y)) k_{t-s}(y)\eta(dy), \quad t > 0, f \in B(D)^+. \tag{3.5}$$

As in Li (1996) one may see that

$$\int_{M(D)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu, d\nu) = \exp\Big\{ -\mu(V_t f) - \int_0^t S_r(\kappa, f) dr\Big\}, \quad f \in B(D)^+, \tag{3.6}$$

defines the transition semigroup $(Q_t^\kappa)_{t\geq 0}$ of an immigration process associated with the catalytic branching super ABM with parameters (η, ϕ, κ).

To understand how formula (3.6) arises, observe that if $u > 0$, then κ_u is a finite measure on D, so

$$\int_{M(D)} \mathrm{e}^{-\nu(f)} Q_t^{\kappa_u}(\mu, d\nu) = \exp\Big\{ -\mu(V_t f) - \int_0^t \kappa_u(V_r f) dr\Big\}, \quad f \in B(D)^+, \tag{3.7}$$

defines the transition semigroup of a catalytic branching super ABM process with immigration; see e.g. Dawson (1993, page 114) or Li (1992). Since

$$\begin{aligned}
\kappa_u(V_r f) &= \kappa_u(P_r f) - \int_D \kappa_u(dx) \int_0^r ds \int_D \phi(y, V_s f(y)) p_{r-s}(x,y)\eta(dy) \\
&= \kappa_{u+r}(f) - \int_0^r ds \int_D \phi(y, V_s f(y)) \kappa_{u+r-s}(x,y)\eta(dy) \\
&\to S_r(\kappa, f)
\end{aligned}$$

as $u \to 0^+$, (3.6) is actually the limit form of (3.7).

4 Fluctuation limits of immigration processes

In this section, we study the small branching fluctuation limits of the catalytic branching super ABM with immigration following the lines set up in Gorostiza (1996), Gorostiza and Li (1998, 1999) and Li (1999). These will give a construction for a class of Ornstein-Uhlenbeck processes taking distribution values.

Let ρ be a strictly positive, infinitely differentiable function on D with $\rho(x) = x$ for $0 < x \leq 1$ and $\rho(x) = x^{-2}$ for $x \geq 2$. Let $B_\rho(D)$ denote the set of functions $f \in B(D)$ satisfying $|f| \leq \mathrm{const} \cdot \rho$. Let $\mathcal{S}(D)$ be the space of infinitely differentiable functions $f \in B(D)$ such that

$$|||f|||_n := \max_{0\leq k\leq n} \sup_{-\infty<u<\infty} \left|(1+u^2)^n \frac{d^k}{du^k} f(\mathrm{e}^u)\right| < \infty, \quad n = 0, 1, 2, \ldots. \tag{4.1}$$

Then $\mathcal{S}(D)$ topologized by the norms $\{|||\cdot|||_n : n = 0, 1, 2, \ldots\}$ is a nuclear space, which may be obtained by the transformation $u \mapsto \mathrm{e}^u$ from the Schwartz space $\mathcal{S}(\mathbb{R})$ defined in Hida (1980, page 305). Let $\mathcal{S}'(D)$ denote the dual space of $\mathcal{S}(D)$.

Suppose that $\{Y_t : t \geq 0\}$ is an immigration process associated with the catalytic branching super ABM with parameters (η, ϕ, κ). For $\theta > 0$ let $S_\theta(D) = \{\mu - \theta^{-1}\lambda : \mu \in M(D)\}$ and define the $S_1(D)$-valued process $Z = \{Z_t : t \geq 0\}$ by $Z_t = Y_t - \lambda$. Then we have a.s.

$$\begin{aligned} & \boldsymbol{E}[\exp\{-Z_{r+t}(f)\}|Z_s : 0 \leq s \leq r] \\ = \ & \exp\Big\{ - Z_r(V_t f) + \lambda(f - V_t f) - \int_0^t S_u(\kappa, f) du\Big\}. \end{aligned}$$

Note that $\int_0^\infty k_t(x)dt = 1$. Then we have

$$\begin{aligned} \int_0^t S_r(\kappa, f)dr &= \int_0^t \kappa_r(f)dr - \int_0^t dr \int_0^r \eta(k_{r-s}\phi(V_s f))ds \\ &= \lambda(f - P_t f) - \int_0^t ds \int_0^{t-s} \eta(k_r\phi(V_s f))dr \\ &= \lambda(f - P_t f) - \int_0^t ds \int_D \Big[1 - \int_D p_{t-s}(x, y)dx\Big]\phi(y, V_s f(y))\eta(dy) \\ &= \lambda(f - V_t f) - \int_0^t \eta(\phi(V_s f))ds. \end{aligned}$$

Therefore, $\{Z_t : t \geq 0\}$ is a Markov process with transition semigroup $(T_t^\kappa)_{t\geq 0}$ given by

$$\int_{S_1(D)} \mathrm{e}^{-\nu(f)} T_t^\kappa(\mu, d\nu) = \exp\Big\{ - \mu(V_t f) + \int_0^t \eta(\phi(V_s f))ds\Big\}, \quad f \in B_\rho(D)^+. \tag{4.2}$$

For any $\theta > 0$ let $\phi_\theta(x, z) = \phi(x, \theta z)$. Then $\phi_\theta(x, z) \to 0$ as $\theta \to 0$. Suppose that $\{Y_t^\theta : t \geq 0\}$ is an immigration process associated with the catalytic branching super ABM with parameters $(\eta, \phi_\theta, \kappa)$ and $Y_0^\theta = \lambda$. We define the fluctuation process $\{Z_t^\theta : t \geq 0\}$ by

$$Z_t^\theta = \theta^{-1}[Y_t^\theta - \lambda], \qquad t \geq 0. \tag{4.3}$$

We are interested in the limiting behavior of the process $\{Z_t^\theta : t \geq 0\}$ as $\theta \to 0$.

¿From the discussions above, we know that $\{Z_t^\theta : t \geq 0\}$ is a Markov process with $Z_0^\theta = 0$ and with semigroup $(R_t^\theta)_{t\geq 0}$ determined by

$$\int_{S_\theta(D)} \mathrm{e}^{-\nu(f)} R_t^\theta(\mu, d\nu) = \exp\Big\{ - \mu(\theta V_t^\theta(f/\theta)) + \int_0^t \eta(\phi(\theta V_s^\theta(f/\theta)))ds\Big\}, \tag{4.4}$$

where $(V_t^\theta)_{t\geq 0}$ is defined by

$$V_t^\theta f(x) + \int_0^t ds \int_D \phi_\theta(y, V_s^\theta f(y)) p_{t-s}(x,y)\eta(dy) = P_t f(x). \tag{4.5}$$

Lemma 4.1 *If $f_\theta \to f \in B(D)^+$ boundedly as $\theta \to 0$, then $\theta V_t^\theta(f_\theta/\theta) \to P_t f$ boundedly as $\theta \to 0$.*

Proof. By (4.5) we have $V_t^\theta f(x) \leq P_t f(x)$ and hence $\theta V_t^\theta(f_\theta/\theta)(x) \leq P_t f_\theta(x)$ for all $t \geq 0$ and $x \in D$. ¿From (4.5) we have

$$\theta V_t^\theta(f_\theta/\theta)(x) + \int_0^t ds \int_D \theta\phi(y, V_s^\theta(f_\theta/\theta)(y)) p_{t-s}(x,y)\eta(dy) = P_t f_\theta(x).$$

Clearly, the second term on the left hand side goes to zero as $\theta \to 0$, we have $\theta V_t^\theta(f_\theta/\theta) \to P_t f$ boundedly as $\theta \to 0$. □

Theorem 4.1 *The finite-dimensional distributions of $\{Z_t^\theta : t \geq 0\}$ converge as $\theta \to 0$ to those of the $\mathcal{S}'(D)$-valued Markov process $\{Z_t^0 : t \geq 0\}$ with $Z_0^0 = 0$ and with semigroup $(R_t^1)_{t\geq 0}$ determined by*

$$\int_{\mathcal{S}'(D)} \mathrm{e}^{-\nu(f)} R_t^1(\mu, d\nu) = \exp\Big\{ - \mu(P_t f) + \int_0^t \eta(\phi(P_s f))ds\Big\}, \quad f \in \mathcal{S}(D)^+, \tag{4.6}$$

where $\phi(\cdot,\cdot)$ is given by (3.1).

Proof. For $0 \leq t_1 < \ldots < t_n$ and $f_1, \ldots, f_n \in \mathcal{S}(D)^+$ set

$$h_j^{(\theta)} = f_j + V_{t_{j+1}-t_j}^{(\theta)}(f_{j+1} + \cdots + V_{t_n - t_{n-1}}^{(\theta)} f_n),$$

where $V_t^{(\theta)} f(x) = \theta V_t^\theta(f/\theta)(x)$. Using (4.4) inductively we get

$$\boldsymbol{E}\exp\Big\{ - \sum_{j=1}^n Z_{t_j}^\theta(f_j)\Big\} = \exp\Big\{ \sum_{j=1}^n \int_0^{t_j - t_{j-1}} \eta(\phi(V_s^{(\theta)} h_j^{(\theta)}))ds\Big\}. \tag{4.7}$$

By Lemma 4.1 it can be proved inductively that

$$h_j^{(\theta)} \to h_j := f_j + P_{t_{j+1}-t_j}(f_{j+1} + \cdots + P_{t_n - t_{n-1}} f_n)$$

boundedly as $\theta \to 0$. Returning to (4.7) we get

$$\lim_{\theta\to 0} \boldsymbol{E} \exp\Big\{ -\sum_{j=1}^n Z_{t_j}^\theta(f_j)\Big\} = \exp\Big\{ \sum_{j=2}^n \int_0^{t_j-t_{j-1}} \eta(\phi(P_s h_j))ds\Big\}.$$

As in Iscoe (1986), it follows that the finite-dimensional distributions of $\{Z_t^\theta : t \geq 0\}$ converge to those of the $\mathcal{S}'(D)$-valued Markov process $\{Z_t^0 : t \geq 0\}$ with $Z_0^0 = 0$ and with transition semigroup $(R_t)_{t\geq 0}$. □

Theorem 4.2 *Let $\varphi(\cdot,\cdot)$ be a function on $D \times I\!R$ with the representation*

$$\varphi(x,z) = -c(x)z^2 + \int_{I\!R^\circ} (\mathrm{e}^{izu} - 1 - izu)m(x,du), \quad x\in D, z\in I\!R, \tag{4.8}$$

where $c \in B(D)^+$ and $(|u| \wedge |u|^2)m(x,du)$ is a bounded kernel from D to $I\!R^\circ$. Then there is a transition semigroup $(R_t)_{t\geq 0}$ on $\mathcal{S}'(D)$ given by

$$\int_{\mathcal{S}'(D)} \mathrm{e}^{i\nu(f)} R_t(\mu,d\nu) = \exp\Big\{ i\mu(P_t f) + \int_0^t \eta(\varphi(P_s f))ds\Big\}, \quad t\geq 0, f\in \mathcal{S}(D). \tag{4.9}$$

Proof. We first assume that $u^2 m(x,du)$ is a bounded kernel from D to $I\!R^\circ$. Let $\{Z_t^1 : t \geq 0\}$ be an $\mathcal{S}'(D)$-valued Markov process with transition semigroup $(R_t^1)_{t\geq 0}$ given by (4.6) and let $\{Z_t^2 : t\geq 0\}$ be an $\mathcal{S}'(D)$-valued Markov process whose transition semigroup $(R_t^2)_{t\geq 0}$ given by

$$\int_{\mathcal{S}'(D)} \mathrm{e}^{-\nu(f)} R_t^2(\mu,d\nu) = \exp\Big\{ -\mu(P_t f) + \int_0^t \eta(\psi(P_s f))ds\Big\}, \quad f\in \mathcal{S}(D)^+,$$

where

$$\psi(x,z) = \int_{-\infty}^0 (\mathrm{e}^{zu} - 1 - zu)m(x,du), \quad x\in D, z\geq 0.$$

Suppose that $\{Z_t^1 : t\geq 0\}$ and $\{Z_t^2 : t\geq 0\}$ are independent. Let $Z_t = Z_t^1 - Z_t^2$ for $t\geq 0$. It is easy to check that $\{Z_t : t \geq 0\}$ is a $\mathcal{S}'(D)$-valued Markov process with transition semigroup $(R_t)_{t\geq 0}$ given by (4.9). In particular, we have proved the existence of the transition semigroup $(R_t)_{t\geq 0}$ under the stronger integral condition on $m(x,du)$. The condition can be relaxed as stated in the theorem by considering a suitable increasing sequence of such kernels and taking the limit in an obvious way. □

Theorem 4.3 *Let $\varphi(\cdot)$ be a function on $I\!R$ given by*

$$\varphi(z) = -cz^2 + \int_{I\!R^\circ} \left(\mathrm{e}^{iuz} - 1 - iuz\right) m(du), \qquad z\in I\!R, \tag{4.10}$$

where $c\geq 0$ and $(|u|\wedge |u|^2)m(du)$ is a finite measure on $I\!R^\circ$. Then there is a transition semigroup $(R_t')_{t\geq 0}$ on $\mathcal{S}'(D)$ given by

$$\int_{\mathcal{S}'(D)} \mathrm{e}^{i\nu(f)} R_t'(\mu,d\nu) = \exp\Big\{ i\mu(P_t f) + \int_0^t \varphi(\kappa_s(f))ds\Big\}, \quad t\geq 0, f\in \mathcal{S}(D). \tag{4.11}$$

Proof. This transition semigroup is obtained from the one in the last theorem by replacing $\varphi(x,z)$ and $\eta(dx)$ in (4.9) by $\varphi(nz)$ and $\delta_{1/2n}(dx)$, respectively, and letting $n \to \infty$. $\square$

5 Ornstein-Uhlenbeck processes with function values

In this section, we show that, under slightly stronger conditions, the Ornstein-Uhlenbeck processes constructed in the last section take function values from $L^2(D, \lambda)$.

Suppose that η is a finite measure on D and $\varphi(\cdot,\cdot)$ is given by (4.8) with $u^2 m(x, du)$ being a bounded kernel from D to $I\!R^\circ$. Let $W(ds, dx)$ be a white noise on $[0,\infty) \times D$ with covariance measure $2c(x)ds\eta(dx)$ and let $N(ds, du, dx)$ be a Poisson random measure on $[0,\infty) \times I\!R^\circ \times D$ with intensity $dsm(x, du)\eta(dx)$. Suppose that $W(ds, dx)$ and $N(ds, du, dx)$ are defined on some complete probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ and are independent of each other. Let $\tilde{N}(ds, du, dx) = N(ds, du, dx) - dsm(x, du)\eta(dx)$. Then we have

Theorem 5.1 *For each $t \geq 0$, the function*

$$Z_t^0(\omega, y) := \int_0^t \int_D p_{t-s}(x,y) W(\omega, ds, dx) + \int_0^t \int_{I\!R^\circ} \int_D u p_{t-s}(x,y) \tilde{N}(\omega, ds, du, dx) \quad (5.1)$$

is well-defined in the $L^2(\Omega \times D, \boldsymbol{P} \times \lambda)$ sense and $\{Z_t^0 : t \geq 0\}$ is a Markov process with state space $L^2(D, \lambda)$, initial value zero and transition semigroup $(R_t)_{t\geq 0}$ given by

$$\int_{L^2(D,\lambda)} \mathrm{e}^{i\langle h,f\rangle} R_t(g, dh) = \exp\Big\{ i\langle g, P_t f\rangle + \int_0^t \eta(\varphi(P_s f)) ds \Big\}, \quad f \in L^2(D, \lambda). \quad (5.2)$$

Moreover, $Z_t^0(\omega, y)$ can be chosen as a function of (t, ω, y) belonging to $L^2([0,\infty) \times \Omega \times D, \lambda \times \boldsymbol{P} \times \lambda)$.

Proof. By the inequality

$$\int_D p_{t-s}(x,y)^2 dy < \frac{1}{2\pi(t-s)} \int_{I\!R} \exp\Big\{ -\frac{y^2}{(t-s)} \Big\} dy = \frac{1}{2\sqrt{\pi(t-s)}},$$

we have

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{ \Big(\int_0^t \int_D p_{t-s}(x,y) W(ds, dx) \Big)^2 \Big\} dy \\ = \;& 2 \int_D dy \int_0^t ds \int_D p_{t-s}(x,y)^2 c(x) \eta(dx) \\ \leq \;& \int_0^t \frac{1}{\sqrt{\pi(t-s)}} ds \int_D c(x) \eta(dx) \\ < \;& \infty \end{aligned}$$

and

$$\begin{aligned}
& \int_D \boldsymbol{E}\Big\{\Big(\int_0^t \int_{I\!R^\circ}\int_D u p_{t-s}(x,y)\tilde{N}(ds,du,dx)\Big)^2\Big\}dy \\
= & \int_D dy \int_0^t ds \int_D \eta(dx)\int_{I\!R^\circ} u^2 p_{t-s}(x,y)^2 m(x,du) \\
\leq & \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} ds \int_D \eta(dx)\int_{I\!R^\circ} u^2 m(x,du) \\
< & \ \infty.
\end{aligned}$$

Then the right hand side of (5.1) is well-defined in the $L^2(\Omega\times D, \boldsymbol{P}\times\lambda)$ sense. By the same reasoning, we see that it is also well-defined in the $L^2([0,\infty)\times\Omega\times D, \lambda\times\boldsymbol{P}\times\lambda)$ sense. For any $f\in L^2(D,\lambda)$, we have

$$\boldsymbol{E}\exp\Big\{i\int_0^t\int_D P_{t-s}f(x)W(ds,dx)\Big\} = \exp\Big\{-\int_0^t ds\int_D c(x)[P_{t-s}f(x)]^2\eta(dx)\Big\}$$

and

$$\begin{aligned}
& \boldsymbol{E}\exp\Big\{i\int_0^t\int_{I\!R^\circ}\int_D uP_{t-s}f(x)\tilde{N}(ds,du,dx)\Big\} \\
= & \exp\Big\{\int_0^t ds\int_D c(x)\eta(dx)\int_{I\!R^\circ}\left(\exp\{iuP_{t-s}f(x)\} - 1 - iuP_{t-s}f(x)\right)m(x,du)\Big\}.
\end{aligned}$$

Thus $\{Z_t^0 : t\geq 0\}$ has the asserted one-dimensional distributions. If $g\in L^2(D,\lambda)$, then $P_t g\in L^2(D,\lambda)$ for all $t\geq 0$. Clearly, the distribution $R_t(g,\cdot)$ of $P_t g + Z_t^0$ has characteristic functional given by (5.2) and $(R_t)_{t\geq 0}$ is a transition semigroup on $L^2(D,\lambda)$. The Markov property of $\{Z_t^0 : t\geq 0\}$ follows by a similar calculation as the above. □

Suppose that $\varphi(\cdot)$ is given by (4.10) with $u^2 m(du)$ being a finite measure on $I\!R^\circ$. Let $\gamma(dx) = (1-e^{-x^2})dx$ for $x\in D$. Let $\{B(t) : t\geq 0\}$ be a one-dimensional Brownian motion with increasing process $2ct$ and let $N(ds,du)$ be a Poisson random measure on $[0,\infty)\times I\!R^\circ$ with intensity $ds m(du)$. Suppose that $\{B(t) : t\geq 0\}$ and $N(ds,du)$ are defined on some complete probability space $(\Omega,\mathcal{F},\boldsymbol{P})$ and are independent of each other. Let $\tilde{N}(ds,du) = N(ds,du) - ds m(du)$. Then we have

Theorem 5.2 *For each $t\geq 0$, the function*

$$Z_t^0(\omega,y) := \int_0^t k_{t-s}(y)B(\omega,ds) + \int_0^t\int_{I\!R^\circ} uk_{t-s}(y)\tilde{N}(\omega,ds,du) \tag{5.3}$$

is well-defined in the $L^2(\Omega\times D,\boldsymbol{P}\times\gamma)$ sense and $\{Z_t^0 : t\geq 0\}$ can be regarded as a Markov process with state space $\mathcal{S}'(I\!R)$, initial value zero and transition semigroup $(R_t')_{t\geq 0}$ given by (4.11). Moreover, $Z_t^0(\omega,y)$ can be chosen as a function of (t,ω,y) belonging to $L^2([0,\infty)\times\Omega\times D,\lambda\times\boldsymbol{P}\times\gamma)$.

Proof. For any $t > 0$,

$$\int_0^t k_s(y)^2 ds \le \int_0^\infty \frac{y^2}{2\pi s^3} e^{-y^2/s} ds = \frac{1}{2\pi y^2}.$$

Then we have

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{\Big(\int_0^t k_{t-s}(y)B(ds)\Big)^2\Big\}\gamma(dy) \\ = \ & 2c\int_D \gamma(dy)\int_0^t k_{t-s}(y)^2 ds \\ < \ & \infty \end{aligned}$$

and

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{\Big(\int_0^t \int_{I\!R^\circ} u k_{t-s}(y)\tilde{N}(ds,du)\Big)^2\Big\}\gamma(dy) \\ = \ & \int_D \gamma(dy)\int_0^t k_{t-s}(y)^2 ds \int_{I\!R^\circ} u^2 m(du) \\ < \ & \infty. \end{aligned}$$

Thus the right hand side of (5.3) is well-defined in the $L^2(\Omega\times D, \boldsymbol{P}\times\gamma)$ sense. Clearly, it is also well-defined in the $L^2([0,\infty)\times\Omega\times D, \lambda\times\boldsymbol{P}\times\gamma)$ sense. For any $f \in \mathcal{S}(I\!R)$, we have

$$\boldsymbol{E}\exp\Big\{i\int_0^t \langle k_{t-s}, f\rangle B(ds)\Big\} = \exp\Big\{-\int_0^t c\langle k_{t-s}, f\rangle^2 ds\Big\}$$

and

$$\begin{aligned} & \boldsymbol{E}\exp\Big\{i\int_0^t \int_{I\!R^\circ} u\langle k_{t-s}, f\rangle \tilde{N}(ds,du)\Big\} \\ = \ & \exp\Big\{\int_0^t ds\int_{I\!R^\circ} \left(\exp\{iu\langle k_{t-s}, f\rangle\} - 1 - iu\langle k_{t-s}, f\rangle\right) m(du)\Big\}. \end{aligned}$$

Thus $\{Z_t^0 : t \ge 0\}$ has the right one-dimensional distributions. The asserted Markov property follows by a similar calculation. □

Theorem 5.3 *If $(1\vee|u|)m(du)$ is a finite measure on $I\!R^\circ$, then for each $t \ge 0$ the function*

$$Z_t^0(y) := \int_0^t \int_{I\!R^\circ} u k_{t-s}(y)\tilde{N}(ds,du) \tag{5.4}$$

belongs to $L^2(D,\lambda)$ a.s. and $\{Z_t^0 : t \ge 0\}$ is a Markov process with state space $L^2(D,\lambda)$, initial value zero and transition semigroup $(R_t')_{t\ge0}$ given by

$$\int_{L^2(D,\lambda)} \mathrm{e}^{i\langle h,f\rangle} R_t'(g,dh) = \exp\Big\{i\langle g, P_t f\rangle + \int_0^t \varphi(\langle k_s, f\rangle)ds\Big\}, \quad f \in L^2(D,\lambda), \tag{5.5}$$

where $\varphi(\cdot)$ is given by (4.10) with $c = 0$.

Proof. For any $t > 0$, we have

$$\int_0^t k_s(y)ds = \int_{y^2/2t}^{\infty} \frac{1}{\sqrt{\pi u}} e^{-u} du,$$

which is bounded in $y \geq 0$ and dominated by

$$\int_{y^2/2t}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u} du = \frac{1}{\sqrt{\pi}} e^{-y^2/2t}$$

for $y \geq \sqrt{t}$. Therefore,

$$\int_0^t ds \int_{I\!R^\circ} uk_{t-s} m(du)$$

belongs to $L^2(D, \lambda)$ under our assumption. Since $k_t \in L^2(D, \lambda)$ for every $t > 0$ and a.s.

$$\int_0^t \int_{I\!R^\circ} uk_{t-s} N(ds, du)$$

is a finite sum, we have $Z_t^0 \in L^2(D, \lambda)$ a.s. If $g \in L^2(D, \lambda)$, then $P_t g \in L^2(D, \lambda)$ for all $t \geq 0$ and the distribution $R_t(g, \cdot)$ of $P_t g + Z_t^0$ has characteristic functional given by (5.5). Clearly, $(R_t)_{t \geq 0}$ is a transition semigroup on $L^2(D, \lambda)$. The Markov property of $\{Z_t^0 : t \geq 0\}$ follows by a standard calculation. □

As in Li (1996) one may see that the Ornstein-Uhlenbeck processes given by (5.2) and (5.5) usually do not have right continuous sample paths, neither do they have the strong Markov property. We shall prove in the next section that they do have those properties if we regard them as processes in another suitably chosen state space.

6 Ornstein-Uhlenbeck processes with signed-measure values

In this section, we show that some of the Ornstein-Uhlenbeck processes given by (5.2) and (5.5) behave very regularly in the space of signed-measures. Indeed, from the proof of Theorem 6.1 we know that they are essentially special forms of the immigration processes studied in Li (1995/1996, 1996).

Given a locally compact metric space E, let $M(E)$ denote the space of finite Borel measures on E. Let $\{f_n\}_{n=1}^{\infty}$ be a dense subset of the space of all bounded uniformly continuous functions on E. We may define a metric $r(\cdot, \cdot)$ on $M(E)$ by

$$r(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n}(1 \wedge |\mu(f_n) - \nu(f_n)|), \qquad \mu, \nu \in M(E). \tag{6.1}$$

Clearly, this metric is compatible with the topology of weak convergence in $M(E)$. Let $S(E) = \{\mu' - \mu'' : \mu', \mu'' \in M(E)\}$ be the space of finite signed-measures on E. Define a metric $\rho(\cdot,\cdot)$ on $S(E)$ by

$$\begin{aligned}\rho(\mu,\nu) &= \inf\{r(\mu',\nu') + r(\mu'',\nu'') : \mu',\mu'',\nu',\nu'' \in M(E) \text{ with} \\ &\qquad \mu' - \mu'' = \mu \text{ and } \nu' - \nu'' = \nu\}.\end{aligned} \tag{6.2}$$

Then $\mu_n \to \mu_0$ in $S(E)$ if and only if there are decompositions $\mu_n = \mu_n' - \mu_n''$ and $\mu_0 = \mu_0' - \mu_0''$ such that $\mu_n' \to \mu_0'$ and $\mu_n'' \to \mu_0''$ in $M(E)$. Below, we shall consider the metric space $(S(E), \rho)$ for $E = (0,\infty)$ or $[0,\infty)$.

Suppose that η is a finite measure and $\varphi(\cdot,\cdot)$ is given by (4.8) with $c(x) \equiv 0$ and with $|u|m(x,du)$ being a bounded kernel from D to $I\!R^\circ$. Suppose that $N(ds,du,dx)$ is a Poisson random measure on $[0,\infty) \times I\!R^\circ \times D$ with intensity $ds m(x,du)\eta(dx)$. Let $\tilde{N}(ds,du,dx) = N(ds,du,dx) - ds m(x,du)\eta(dx)$. Let

$$Y_t(f) := \int_0^t \int_{I\!R^\circ} \int_D u P_{t-s} f(x) \tilde{N}(ds,du,dx), \qquad t \geq 0, f \in B(D). \tag{6.3}$$

Theorem 6.1 *The process $\{Y_t : t \geq 0\}$ defined by (6.3) is an a.s. right continuous $S(D)$-valued strong Markov process with transition semigroup $(R_t)_{t\geq 0}$ defined by*

$$\int_{S(D)} \mathrm{e}^{i\nu(f)} R_t(\mu, d\nu) = \exp\Big\{ i\mu(P_t f) + \int_0^t \eta(\varphi(P_s f)) ds \Big\}, \quad f \in B(D). \tag{6.4}$$

Proof. We define the positive part $\{Y_t' : t \geq 0\}$ of $\{Y_t : t \geq 0\}$ by

$$Y_t'(f) := \int_0^t \int_0^\infty \int_D u P_{t-s} f(x) N(ds,du,dx), \qquad t \geq 0, f \in B(D).$$

By the assumptions,

$$\begin{aligned}\boldsymbol{E}\{Y_t'(f)\} &= \int_0^t ds \int_D \eta(dx) \int_0^\infty u P_{t-s} f(x) m(x,du) \\ &\leq t\|f\|\eta(D) \sup_{x\in D} \int_0^\infty u m(x,du) \\ &< \infty.\end{aligned}$$

Then $\{Y_t' : t \geq 0\}$ is a well-defined $M(D)$-valued process, which is clearly a special form of the immigration process considered in Li (1996) without branching. By Li (1996, Theorem 4.1), $\{Y_t' : t \geq 0\}$ is a.s. right continuous. Similarly, the negative part $\{Y_t'' : t \geq 0\}$ of $\{Y_t : t \geq 0\}$ defined by

$$Y_t''(f) := -\int_0^t \int_{-\infty}^0 \int_D u P_{t-s} f(x) N(ds,du,dx), \qquad t \geq 0, f \in B(D),$$

is also an a.s. right continuous immigration process. Then one can easily see that $\{Y_t : t \geq 0\}$ defined by (6.3) is an a.s. right continuous $S(D)$-valued Markov process with transition semigroup $(R_t)_{t\geq 0}$. The strong Markov property holds since $(R_t)_{t\geq 0}$ is clearly Feller. □

Suppose that $\varphi(\cdot)$ is given by (4.10) with $c = 0$ and with $|u|m(du)$ being a finite measure on $\mathbb{R}^\circ$. Suppose that $N(ds, du)$ is a Poisson random measure on $[0, \infty) \times \mathbb{R}^\circ$ with intensity $dsm(du)$. Let $\tilde{N}(ds, du) = N(ds, du) - dsm(du)$. Let

$$Y_t(f) := \int_0^t \int_{\mathbb{R}^\circ} u\kappa_{t-s}(f)\tilde{N}(ds, du), \qquad t \geq 0, f \in B(D). \tag{6.5}$$

By a similar argument as in the proof of Theorem 6.1 we get

Theorem 6.2 *The process $\{Y_t : t \geq 0\}$ defined by (6.5) is an $S(D)$-valued Markov process with transition semigroup $(R'_t)_{t\geq 0}$ defined by*

$$\int_{S(D)} \mathrm{e}^{i\nu(f)} R'_t(\mu, d\nu) = \exp\Big\{i\mu(P_t f) + \int_0^t \varphi(\kappa_s(f))ds\Big\}, \quad f \in B(D). \tag{6.6}$$

As in Li (1996) one can see that the process (6.5) does not have right continuous modification. Observe that $h(x) := (1 - e^{-x})$ is an excessive function of $(P_t)_{t\geq 0}$ and

$$T_t f(x) = \begin{cases} h(x)^{-1}P_t(hf)(x) & \text{for } t > 0 \text{ and } x > 0, \\ 2\kappa_t(hf) = (d/dx)P_t(hf)(0^+) & \text{for } t > 0 \text{ and } x = 0, \end{cases} \tag{6.7}$$

defines the transition semigroup $(T_t)_{t\geq 0}$ of a Markov process on $[0, \infty)$.

Theorem 6.3 *Let $\{Y_t : t \geq 0\}$ be defined by (6.5), and let $Z_t(\{0\}) = 0$ and $Z_t(dx) = (1 - e^{-x})Y_t(dx)$ for $x > 0$. Then $\{Z_t : t \geq 0\}$ is an $S([0, \infty))$-valued Markov process with transition semigroup $(S_t)_{t\geq 0}$ defined by*

$$\int_{S([0,\infty))} \mathrm{e}^{i\nu(f)} S_t(\mu, d\nu) = \exp\Big\{i\mu(T_t f) + \int_0^t \varphi(\kappa_s(hf))ds\Big\}, \quad f \in B([0, \infty)). \tag{6.8}$$

Moreover, $\{Z_t : t \geq 0\}$ has a right continuous strong Markov realization.

Proof. The first assertion holds by Theorem 6.2. Observe that

$$\int_{S([0,\infty))} \mathrm{e}^{i\nu(f)} S_t(\mu, d\nu) = \exp\Big\{i\mu(T_t f) + \int_0^t \varphi(2^{-1}T_s f(0))ds\Big\}, \quad f \in B([0, \infty)),$$

by (6.7) and (6.8). Then the second assertion follows from Li (1996, Theorem 4.1) as in the proof of Theorem 6.1. □

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