Quasi-regular Dirichlet forms: Examples and counterexamples

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ABSTRACT

We prove some new results on quasi-regular Dirichlet forms. These include results on perturbations of Dirichlet forms, change of speed measure, and tightness. The tightness implies the existence of an associated right continuous strong Markov process. We also discuss applications to a number of examples including cases with possibly degenerate (sub)-elliptic part, diffusions on loops spaces, and certain Fleming-Viot processes.

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0. Introduction.

The purpose of this paper is to bring together some new results on quasi-regular Dirichlet forms that were obtained recently. In Section 1 we start with some examples of semi-Dirichlet forms on an open subset of $\mathbb{R}^d$ with possibly degenerate (sub)-elliptic part. Our treatment of these forms extends some of the results in [Str 88]. Subsequently, we consider perturbations of Dirichlet forms by smooth measures, along the lines of [AM 91b], and also look at the effect of changing the underlying speed measure (cf. Section 2). In Section 3 we extend our earlier results on tightness to a more general class of Dirichlet forms which consist of a "square field operator"-type form perturbed by a jump and killing term. As a consequence one can construct an associated (special) standard process on the basis of the general theory in [MR 92]. We give several applications in Section 4, i.e., construct diffusions on Banach spaces and loop spaces, and also construct certain Fleming-Viot processes (which are measure-valued). We note that in the Section 1 we look at semi-Dirichlet forms, but afterwards we restrict ourselves to Dirichlet forms (see Definition 0.3 below for the difference). The reason is that we will sometimes use Ancona's result (see Remark 0.4) and it is not known if this result extends to semi-Dirichlet forms. Many of the results for Dirichlet forms do carry over to semi-Dirichlet forms, we refer the interested reader to [MOR 93].

Until recently, the general theory of Dirichlet forms had been restricted to the case where the underlying space is locally compact. In M. Fukushima's book [F 80], which is the standard reference in the area, the local compactness is used throughout and is crucial in the construction of the associated Markov processes. Fukushima assumes that E is a locally compact, separable metric space and that m is a positive Radon measure on $\mathcal{B}(E)$ with full support. He then constructs a Markov process, indeed a Hunt process, associated to any *regular* Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ (cf. below for definitions) where regularity means

$$D(\mathcal{E}) \cap C_0(E) \text{ is } \mathcal{E}_1^{1/2}\text{-dense in } D(\mathcal{E}), \text{ and is uniformly dense in } C_0(E). \tag{0.1}$$

Here $C_0(E)$ is the space of continuous real-valued functions with compact support.

Now the local compactness assumption, of course, eliminates the possibility of using Fukushima's theory in the study of infinite-dimensional processes. Nevertheless, in the years since the publication of [F 80] several authors (cf. eg. [AH-K 75, 77 a,b], [Ku 82], [AR 89,91], [S 90] and see also the reference list in [MR 92]) have been able to modify Fukushima's construction in special cases and obtain processes in infinite-dimensional state spaces. Recently a more general framework in which such constructions are possible has been developed. This is the theory of (non-symmetric) *quasi-regular* Dirichlet forms, which are defined below. The fundamental existence result in this framework is found in [MR 92; Chapter IV, Theorem 6.7] and it says the following:

Theorem 0.1. *Let E be a metrizable Lusin space. Then a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is quasi-regular if and only if there exists a pair $(\mathbf{M}, \widehat{\mathbf{M}})$ of normal, right continuous, strong Markov processes associated with $(\mathcal{E}, D(\mathcal{E}))$.*

This says that the class of quasi-regular Dirichlet forms is the correct setting for the study of those forms associated with nice Markov processes. Z.M. Ma, L. Overbeck, and M. Röckner [MOR 93] have recently proved a one-sided version of the existence result for quasi-regular *semi*-Dirichlet forms; see Definition 0.3 below. In this case we do not get a pair of processes but only the process $\mathbf{M}$.

In order to explain what a quasi-regular Dirichlet form is we first need some preparation. For a detailed exposition we refer the reader to [AMR 93a] and, in particular, to the monograph [MR 92].

Let E be a Hausdorff topological space, and $\mathcal{B}(E)$ be the Borel sets in E. Fix a positive, σ-finite measure m on $\mathcal{B}(E)$.

Definition 0.2. A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *coercive closed form* on (real) $L^2(E;m)$ if $D(\mathcal{E})$ is a dense linear subspace of $L^2(E;m)$ and if $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to I\!R$ is a bilinear form such that the following conditions hold:

(i) $\mathcal{E}(u,u) \geq 0$ for all $u \in D(\mathcal{E})$.

(ii) $D(\mathcal{E})$ is a Hilbert space when equipped with the inner product $\tilde{\mathcal{E}}_1(u,v) := (1/2)\{\mathcal{E}(u,v) + \mathcal{E}(v,u)\} + (u,v)_{L^2(E;m)}$.

(iii) $(\mathcal{E}_1, D(\mathcal{E}))$ satisfies the sector condition, i.e., there exists a constant $K > 0$ such that $|\mathcal{E}_1(u,v)| \leq K\ \mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}$, for all $u, v \in D(\mathcal{E})$.

Here and henceforth, $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{L^2(E;m)}$ for $\alpha \geq 0$, and $\mathcal{E}(u) := \mathcal{E}(u,u)$. For the one-to-one correspondence between coercive closed forms, their generators, resolvents, and semigroups we refer to [MR 92; Chapter I].

Definition 0.3. A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called a *semi-Dirichlet form* (cf. [CaMe 75], [MOR 93]) if it has the following (unit) contraction property: for all $u \in D(\mathcal{E})$, we have $u^+ \wedge 1 \in D(\mathcal{E})$ and

$$\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0. \tag{0.2}$$

If, in addition, $\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0$, then $(\mathcal{E}, D(\mathcal{E}))$ is called a *Dirichlet form.*

Remark 0.4. If $\psi : I\!R \to I\!R$ satisfies $\psi(0) = 0$ and $|\psi(t) - \psi(s)| \leq |t - s|$ for all $t, s \in I\!R$, then ψ is called a *normal contraction.* Ancona [An 76] has shown that if $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form and ψ is a normal contraction, then the mapping $u \to \psi(u)$ is strongly continuous on the Hilbert space $(D(\mathcal{E}), \tilde{\mathcal{E}}_1)$. It follows easily that this conclusion also holds if ψ is a function with a bounded first derivative and $\psi(0) = 0$.

Definition 0.5. Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E;m)$.

(i) For a closed subset $F \subseteq E$ we define

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E})|\ \ u = 0\ \ m\text{-a.e. on } E \setminus F\}. \tag{0.3}$$

Note that $D(\mathcal{E})_F$ is a closed subspace of $D(\mathcal{E})$.

(ii) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$-*nest* if $\cup_{k\geq 1} D(\mathcal{E})_{F_k}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$.

(iii) A subset $N \subset E$ is called $\mathcal{E}$-*exceptional* if $N \subseteq \cap_{k\geq 1} F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$. A property of points in E holds $\mathcal{E}$-*quasi-everywhere* (abbreviated $\mathcal{E}$-q.e.), if the property holds outside some $\mathcal{E}$-exceptional set. It can be seen that every $\mathcal{E}$-exceptional set has m-measure zero.

(iv) An $\mathcal{E}$-q.e. defined function $f : E \to \mathbb{R}$ is called $\mathcal{E}$-*quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ so that $f|_{F_k}$ is continuous for each $k \in \mathbb{N}$.

(v) Let $f, f_n,\ n \in \mathbb{N}$, be $\mathcal{E}$-q.e. defined functions on E. We say that $(f_n)_{n\in\mathbb{N}}$ converges $\mathcal{E}$-*quasi-uniformly* to f if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f_n \to f$ uniformly on each F_k.

We shall use the following result throughout this paper (cf. [MR 92; Chapter III, Proposition 3.5] and [MOR; Proposition 2.18]).

Lemma 0.6. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E;m)$. Let $u_n \in D(\mathcal{E})$, which have $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n, n \in \mathbb{N}$, such that $u_n \to u \in D(\mathcal{E})$ with respect to $\tilde{\mathcal{E}}_1^{1/2}$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ of u so that $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ converges $\mathcal{E}$-quasi-uniformly to $\tilde{u}$.*

We are now able to define a quasi-regular semi-Dirichlet form.

Definition 0.7. A semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called *quasi-regular* if:

(QR1) There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(QR2) There exists an $\tilde{\mathcal{E}}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(QR3) There exist $u_n \in D(\mathcal{E}), n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n, n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Remark 0.8. Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form.

(i) By [MOR 93; Proposition 3.6] (cf. [MR 92; Chapter IV, Remark 3.2 (iii)]) the compact sets F_k in (QR1) can always be chosen to be metrizable.

(ii) (QR2) implies that every $u \in D(\mathcal{E})$ has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$. Henceforth, for any subset D of $D(\mathcal{E})$ we will use $\tilde{D}$ to denote the set of all $\mathcal{E}$-quasi-continuous m-versions of elements of D. That is, $\tilde{D} = \{\tilde{u} \,|\, u \in D\}$.

(iii) By [MOR 93; Proposition 2.18(ii)] (cf. [MR 92; Chapter III, Proposition 3.6]) there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $m(F_k) < \infty$ for each k. Using this and [Bou 74; Chapter IX, Sect. 6, Def. 9, Théorème 6 and Proposition 10] we can prove that m is inner regular on $\mathcal{B}(E)$, i.e., $m(B) = \sup\{m(K) \,|\, K \subseteq B \text{ and } K \text{ is compact}\}$ for all

$B \in \mathcal{B}(E)$.

(iv) We define the symmetric part of $(\mathcal{E}, D(\mathcal{E}))$ by setting $\tilde{\mathcal{E}}(u,v) := 1/2\{\mathcal{E}(u,v) + \mathcal{E}(v,u)\}$ for $u, v \in D(\mathcal{E})$. If $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, then $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is also a Dirichlet form. We notice that the definitions of $\mathcal{E}$-nest, $\mathcal{E}$-quasi-continuity and quasi-regularity only depend on $\mathcal{E}$ through its symmetric part $\tilde{\mathcal{E}}$.

(v) The property (QR1) is equivalent (see [MR92; Chapter III, Theorem 2.11] and [MOR 93; Theorem 2.14]) to the *tightness* of an associated capacity and is absolutely vital in the construction of an associated Markov process (see [LR 92], [RS 92], [MOR 93]). In this paper we will not use the notion of capacity, instead we will stick with the equivalent "nest" formulation.

The new concept of a quasi-regular semi-Dirichlet form includes the classical concept of a regular semi-Dirichlet form, this follows from the next proposition (cf. [MR 92; Chapter IV, Example 4a]). We repeat the proof here for the convenience of the reader.

Proposition 0.9. *Assume E is a locally compact, separable, metric space and m is a positive Radon measure on $\mathcal{B}(E)$. If $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E;m)$ (see (0.1)), then $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.*

Proof. We only show (QR1), as (QR2) and (QR3) are easy exercises. By the topological assumptions on E, we may write $E = \cup_{k=1}^{\infty} F_k$, where $(F_k)_{k \in \mathbb{N}}$ is an increasing sequence of compact sets in E so that F_k is contained in the interior of F_{k+1} for all $k \geq 1$. It is then easy to see that

$$C_0(E) \cap D(\mathcal{E}) \subseteq \cup_{k=1}^{\infty} D(\mathcal{E})_{F_k}, \tag{0.4}$$

which concludes the proof. □

Remark 0.10. The first example in Section 5 is a Dirichlet form that satisfies (0.1), but is not quasi-regular. The space E is this example is a separable, compact, non-metrizable Hausdorff space.

1. Degenerate semi-Dirichlet forms in finite dimensions.

The purpose of this section is to generalize the standard class of examples of semi-Dirichlet forms on an open (not necessarily bounded) set $U \subset I\!R^d$, $d \geq 3$ (cf. [MR 92; Chapter II, Subsection 2d)]). In particular, we want to allow sub-elliptic, possibly degenerate diffusion parts. We need some preparations. We adopt the terminology of Chapters I and II from [MR 92].

Let $\sigma, \rho \in L^1_{\text{loc}}(U; dx)$, $\sigma, \rho > 0$ $dx-$a.e. where dx denotes Lebesgue measure. The following symmetric form will serve as a "reference form". Set for $u, v \in C_0^\infty(U)$ (:= all infinitely differentiable functions with compact support in U)

$$\mathcal{E}_\rho(u,v) = \sum_{i,j=1}^d \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\rho\, dx. \tag{1.1}$$

Assume that

$$(\mathcal{E}_\rho, C_0^\infty(U)) \text{ is closable on } L^2(U;\sigma dx). \tag{1.2}$$

Remark 1.1. A sufficient condition for (1.2) to hold is that ρ, σ satisfy *Hamza's condition* (see [MR 92; Chapter II, Subsection 2a)]). We recall that a $\mathcal{B}(U)$-measurable function $f : U \to [0, \infty)$ satisfies *Hamza's condition* if for dx-a.e. $x \in U$, $f(x) > 0$ implies that for some $\epsilon > 0$

$$\int_{\{y:\|y-x\|\leq\epsilon\}} (f(y))^{-1}\, dy < \infty, \tag{1.3}$$

where we set $\frac{1}{0} := +\infty$ and $\|\cdot\|$ denotes Euclidean distance in $I\!R^d$. In particular, σ, ρ may have zeros, and (1.2) holds if, for example, σ, ρ are lower semi-continuous. However, there is also a generalized version, a kind of "Hamza condition on rays", which, if it is fulfilled for σ, ρ, also implies (1.2) (cf. [AR 90; (5.7)] and [AR 91; Theorem 2.4]). In particular, if σ, ρ are weakly differentiable then (1.2) holds.

Now let $a_{ij}, b_i, d_i, c \in L^1_{\text{loc}}(U; dx), 1 \leq i, j \leq d$, and define for $u, v \in C_0^\infty(U)$

$$\begin{aligned}\mathcal{E}(u,v) = \sum_{i,j=1}^d \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\, dx + \sum_{i=1}^d \int \frac{\partial u}{\partial x_i} v b_i\, dx \\ + \sum_{i=1}^d \int u\frac{\partial v}{\partial x_i} d_i\, dx + \int uvc\, dx.\end{aligned} \tag{1.4}$$

Then $(\mathcal{E}, C_0^\infty(U))$ is a densely defined bilinear form on $L^2(U; \sigma dx)$. Set $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $\underline{b} := (b_1, \ldots, b_d)$, and $\underline{d} := (d_1, \ldots, d_d)$. We define F to be the set of all functions $g \in L^1_{\text{loc}}(U; dx)$ such that the distributional derivatives $\frac{\partial g}{\partial x_i}$, $1 \leq i \leq d$, are in $L^1_{\text{loc}}(U; dx)$ such that $\|\nabla g\|(g\sigma)^{-1/2} \in L^\infty(U; dx)$ or $\|\nabla g\|^p(g^{p+1}\sigma^{p/q})^{-1/2} \in L^d(U; dx)$ for some $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p < \infty$. We say that a $\mathcal{B}(U)$-measurable function f has property $(A_{\rho,\sigma})$ if one of the following conditions holds:

(i) $f(\rho\sigma)^{-1/2} \in L^\infty(U;dx)$

(ii) $f^p(\rho^{p+1}\sigma^{p/q})^{-1/2} \in L^d(U;dx)$ for some $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p < \infty$, and $\rho \in F$.

Theorem 1.2. *Suppose that*

(1.5) $\|\underline{\xi}\|^2_{\tilde{a}} := \sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \geq \rho\|\underline{\xi}\|^2$ *dx-a.e. for all* $\underline{\xi} = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d$.

(1.6) $\check{a}_{ij}\rho^{-1} \in L^\infty(U;dx)$.

(1.7) For all $K \subset U$*, K compact,* $1_K\|\underline{b}+\underline{d}\|$ *and* $1_K c^{1/2}$ *have property* $(A_{\rho,\sigma})$*, and* $(c+\alpha_0\sigma)dx - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i}$ *is a positive measure on* $\mathcal{B}(U)$ *for some* $\alpha_0 \in (0,\infty)$.

(1.8) $\|\underline{b}-\underline{d}\|$ *has property* $(A_{\rho,\sigma})$.

(1.9) $\underline{b} = \underline{\beta}+\underline{\gamma}$ *such that* $\|\underline{\beta}\|, \|\underline{\gamma}\| \in L^1_{\text{loc}}(U;dx)$*,* $(c+\alpha_0\sigma)dx - \sum_{i=1}^d \frac{\partial \gamma_i}{\partial x_i}$ *is a positive measure on* $\mathcal{B}(U)$ *and* $\|\underline{\beta}\|$ *has property* $(A_{\rho,\sigma})$.

Then:

(i) *There exists* $\alpha \in (0,\infty)$ *such that* $(\mathcal{E}_\alpha, C_0^\infty(U))$ *is closable on* $L^2(U;\sigma dx)$ *and its closure* $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ *is a regular semi-Dirichlet form. In particular, the corresponding semigroup* $(T_t)_{t>0}$ *is* sub-Markovian *and there exists a diffusion process* **M** properly associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ *(cf. [MR 92; Chapter IV]).*

(ii) *If* $\underline{\beta} \equiv 0$ *in (1.9) then* α *can be taken to be* α_0 *as given in (1.7), and* $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ *is a regular Dirichlet form. In particular, both corresponding semigroups* $(T_t)_{t>0}$*,* $(\widehat{T}_t)_{t>0}$ *are sub-Markovian and there exists a pair* $(\mathbf{M}, \widehat{\mathbf{M}})$ *of diffusion processes* properly associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ *(cf. [MR 92; Chapter IV]).*

Remark 1.3.

(i) Theorem 1.2 extends a result obtained by different techniques by D.W. Stroock (cf. [Str 88, Theorem II 3.8]) in the strictly elliptic case ($\rho \equiv$ const) with $\sigma \equiv 1$, $a_{ij} \in L^\infty(U;dx)$, $\check{a}_{ij} \equiv 0$ for $1 \leq i,j \leq d$, $\underline{\gamma}, \underline{d} \equiv 0$, and $\|\underline{\beta}\| \in L^\infty(U;dx)$. We emphasize, however, that Stroock's result in this particular case is stronger than ours since he even proves the corresponding semigroup to be strongly Feller and to have a density with respect to Lebesgue measure.

(ii) The analytic part of the proof of Theorem 1.2 is quite elementary. One of the main ingredients is the classical Sobolev Lemma (cf. [Da 89; Theorem 1.7.1]), i.e., if $\lambda := \frac{2(d-1)}{(d-2)d^{1/2}}$, then

$$\|u\|_q \leq \lambda \,\big\|\, \|\nabla u\| \,\big\|_2 \text{ for all } u \in C_0^\infty(U), \tag{1.10}$$

where $\frac{1}{q} + \frac{1}{d} = \frac{1}{2}$ and for $p \geq 1$, $\| \ \|_p$ denotes the usual norm in $L^p(U;dx)$. A part of the proof of Theorem 1.2 is close to the classical one in [St 65] where $\rho \equiv \sigma \equiv 1$. However even in this case our proof of the Dirichlet property is quite different and shorter (cf. [Bl 71, (10.7)] for the classical proof), and we need less restrictive assumptions on the coefficients (namely, e.g. merely $\|b\|$, $\|d\| \in L^d_{\text{loc}}(U; dx)$, $c \in L^{d/2}_{\text{loc}}(U; dx)$ instead of the global integrability conditions in [St 65]). This is mainly due to our more refined techniques to prove closability which also permit us to take so general σ and ρ.

(iii) We stress that in the situation of Theorem 1.2 we can replace U by a Riemannian manifold M as long as condition (1.10) or more generally the following inequality holds for some $\alpha > 0$

$$\|u\|_{\frac{2d}{d-2}} \leq \text{ const. } \|(-\Delta + \alpha)^{1/2} u\|_2 \text{ for all } u \in C_0^\infty(M), \tag{1.11}$$

where Δ is the Laplacian on M. We refer e.g. to [VSC 92] for examples.

Before we prove Theorem 1.2 we discuss some examples for the function ρ in (1.1), (1.5).

Examples 1.4.

(i) In the case $\sigma \equiv 1$ it is easy to check that our conditions allow ρ to have zeros of order $\|x\|^\alpha$, $\alpha \geq 2$.

(ii) Suppose that $U = I\!R^d$ and that for every $K \subset I\!R^d$, K compact, there exists $c_K \in (0, \infty)$ such that

$$\sum_{i,j=1}^d \tilde{a}_{ij} \xi_i \xi_j \geq c_K \|\xi\|^2 \quad dx\text{-a.e.} \tag{1.12}$$

for all $\underline{\xi} = (\xi_1, \dots, \xi_d) \in I\!R^d$ (i.e., we have *local strict ellipticity*). Then there exists a strictly positive C^1-function ρ satisfying (1.5).

(iii) Suppose ∂U is smooth and let g be a smooth distance function in the sense of [LM 72]. Define

$$\rho := g^\alpha, \ \alpha \geq 2. \tag{1.13}$$

Then it is easy to check that $\|\nabla \rho\| \rho^{-1/2} \in L^\infty(U; dx)$.

For the proof of Theorem 1.2 we need two lemmas.

Lemma 1.5. *Let f be a $\mathcal{B}(U)$-measurable function having property $(A_{\rho,\sigma})$. Then there exist $\delta, \eta \in (0, \infty)$, with δ arbitrarily small, such that for all $u \in C_0^\infty(U)$,*

$$\int f^2 \rho^{-1} u^2 \, dx \leq \delta \int \|\nabla u\|^2 \rho \, dx + \eta \int u^2 \sigma \, dx. \tag{1.14}$$

Proof. The assertion is obviously true in case $(A_{\rho,\sigma})(i)$. In case $(A_{\rho,\sigma})(ii)$ with $p, q \in (1, \infty)$ we have for all $\delta_1 \in (0, 1)$ and $u \in C_0^\infty(U)$ that

$$\int f^2 \rho^{-1} u^2 \, dx \leq \frac{\delta_1^{p/q}}{p} A + \frac{1}{q\delta_1} \int u^2 \sigma \, dx, \tag{1.15}$$

where

$$A := \int f^{2p} \rho^{-p} \sigma^{-p/q} u^2 \, dx \tag{1.16}$$

and where we used that $a^{1/p} b^{1/q} \le a/p + b/q$ for $a, b \in (0, \infty)$. Setting

$$f_0 := f^p (\rho^{p+1} \sigma^{p/q})^{-1/2} \tag{1.17}$$

we obtain by Hölder's inequality with $q \in (2, \infty)$, $\frac{1}{q} + \frac{1}{d} = \frac{1}{2}$, and (1.10) that

$$\begin{aligned} A &\le \|f_0\|_d^2 \, \|\rho^{1/2} u\|_q^2 \\ &\le \lambda^2 \|f_0\|_d^2 \Big(\int \|\rho^{1/2} \nabla u + \frac{1}{2} u \rho^{-1/2} \nabla \rho\|^2 \, dx \Big) \\ &\le \lambda^2 \|f_0\|_d^2 \Big(2 \int \|\nabla u\|^2 \rho \, dx + \frac{B}{2} \Big), \end{aligned} \tag{1.18}$$

where

$$B := \int \|\nabla \rho\|^2 \rho^{-1} u^2 \, dx. \tag{1.19}$$

If $\|\nabla \rho\| (\rho\sigma)^{-1/2}$ is bounded, the assertion obviously follows. If

$$\rho_0 := \|\nabla \rho\|^{p_1} (\rho^{p_1+1} \sigma^{p_1/q_1})^{-1/2} \in L^d(U; dx), \tag{1.20}$$

applying what we have proved so far with $f := \|\nabla \rho\|$ we obtain for all $\delta_2 \in (0,1)$ and $u \in C_0^\infty(U)$ that

$$B \le \frac{\delta_2^{p_1/q_1}}{p_1} \, \|\rho_0\|_d^2 \Big(2 \int \|\nabla u\|^2 \rho \, dx + \frac{B}{2} \Big) + \frac{1}{q_1 \delta_2} \int u^2 \sigma \, dx. \tag{1.21}$$

Solving for B we get for δ_2 small enough that

$$B \le \Big(1 - \frac{\delta_2^{p_1/q_1}}{2p_1} \|\rho_0\|_d^2 \Big)^{-1} \Big(\frac{2\delta_2^{p_1/q_1}}{p_1} \|\rho_0\|_d^2 \int \|\nabla u\|^2 \rho \, dx + \frac{1}{q_1 \delta_2} \int u^2 \sigma \, dx \Big), \tag{1.22}$$

and resubstitution in (1.15) and (1.18) yields the assertion. The case where $\|\nabla \rho\| \rho^{-1} \in L^d(U; dx)$ is similar. If f has property $(A_{\rho,\sigma})(ii)$ with $p = 1$, $q = \infty$, we have as in (1.18), since

$$f = 1_{\{f(\rho\sigma)^{-1/2} \le k\}} f + f_k \tag{1.23}$$

with $f_k := 1_{\{f(\rho\sigma)^{-1/2} > k\}} f$, $k \in \mathbb{N}$, that

$$\int f^2 \rho^{-1} u^2 \, dx \le 2k^2 \int u^2 \sigma \, dx + 2 \|f_k \rho^{-1}\|_d^2 \, \|\rho^{1/2} u\|_q^2. \tag{1.24}$$

Noting that by assumption $\|f_k \rho^{-1}\|_d \to 0$ as $k \to \infty$, we obtain the assertion also in this case by the same arguments as before. □

Let $\langle\, ,\, \rangle$ denote the Euclidean inner product on $\mathbb{R}^d$.

Lemma 1.6. *Consider the situation of Theorem 1.2. Then for any $\epsilon \in (0,1)$ there exists $\alpha \in [\alpha_0, \infty)$ such that for all $u \in C_0^\infty(U)$,*

$$\int |\langle \nabla u, \underline{\beta} \rangle u| \, dx \leq \epsilon \Big(\mathcal{E}_\alpha(u,u) - \int \langle \nabla u, \underline{\beta} \rangle u \, dx \Big). \tag{1.25}$$

Proof. We first note that for all $u \in C_0^\infty(U)$,

$$\begin{aligned} &\mathcal{E}_{\alpha_0}(u,u) - \int \langle \nabla u, \underline{\beta} \rangle u \, dx \\ &\qquad = \int \|\nabla u\|_{\tilde{a}}^2 \, dx + \frac{1}{2} \int [\langle \nabla u^2, \underline{\gamma} + \underline{d} \rangle + 2(c + \alpha_0 \sigma) u^2] \, dx \\ &\qquad \geq \int \|\nabla u\|_{\tilde{a}}^2 \, dx. \end{aligned} \tag{1.26}$$

Furthermore, by (1.5) and Lemma 1.5 for all $\delta', \delta \in (0,1)$ and $u \in C_0^\infty(U)$,

$$\begin{aligned} \int |\langle \nabla u, \underline{\beta} \rangle u| \, dx &\leq \frac{1}{2} \int (\delta' \rho \|\nabla u\|^2 + \frac{1}{\delta'} \|\underline{\beta}\|^2 \rho^{-1} u^2) \, dx \\ &\leq \frac{1}{2}(\delta' + \frac{\delta}{\delta'}) \int \|\nabla u\|_{\tilde{\alpha}}^2 \, dx + \frac{\eta}{2\delta'} \int u^2 \sigma \, dx, \end{aligned} \tag{1.27}$$

for some $\eta \in (0, \infty)$. Now the assertion follows by (1.26). $\square$

Proof of Theorem 1.2. Let ϵ, α be as in Lemma 1.6. Since $\epsilon < 1$ the positive definiteness of $(\mathcal{E}_\alpha, C_0^\infty(U))$ is obvious by 1.7. To prove closability of $(\mathcal{E}_\alpha, C_0^\infty(U))$ on $L^2(U; \sigma dx)$ first note that by Lemma 1.6 for all $u \in C_0^\infty(U)$

$$(1+\epsilon)^{-1} \mathcal{E}_\alpha(u,u) \leq \mathcal{E}_\alpha(u,u) - \int \langle \nabla u, \underline{\beta} \rangle u \, dx \leq (1-\epsilon)^{-1} \mathcal{E}_\alpha(u,u). \tag{1.28}$$

Hence it suffices to consider the case $\underline{\beta} \equiv 0$. By [MR 92; Chapter II, Subsection 2b)] we know that if for $u, v \in C_0^\infty(U)$,

$$\mathcal{E}^{\tilde{a}}(u,v) := \sum_{i,j=1}^d \int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \tilde{a}_{ij} \, dx, \tag{1.29}$$

then $(\mathcal{E}^{\tilde{a}}, C_0^\infty(U))$ is closable on $L^2(U; \sigma dx)$. Let μ be the positive Radon measure on $\mathcal{B}(U)$ defined by

$$\mu := 2(c + \alpha\sigma) dx - \sum_{i=1}^d \frac{\partial (d_i + b_i)}{\partial x_i} \tag{1.30}$$

(cf. (1.7), (1.9) and recall that $\underline{\beta} \equiv 0$).

Claim 1.7.

Let $u_n \in C_0^\infty(U)$, $n \in N$, with $u_n \to 0$ in $L^2(U;\sigma dx)$ as $n \to \infty$, and $\|\nabla u_n\| \to 0$ in $L^2(U;\rho dx)$ as $n \to \infty$. Then there exists a subsequence $(u_{n_k})_{k\in N}$ with $u_{n_k} \to 0$ μ-a.e. as $k \to \infty$.

Before we prove the claim, for the convenience of the reader we repeat the (modified) argument from [MR 92; page 51] that it implies closability. So, let $u_n \in C_0^\infty(U)$, $n \in I\!N$ such that $u_n \to 0$ in $L^2(U;\sigma dx)$ as $n \to \infty$, and $\mathcal{E}_\alpha(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$. Then by (1.26), $\mathcal{E}^{\tilde a}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$, and, since $(\mathcal{E}^{\tilde a}, C_0^\infty(U))$ is closable in $L^2(U;\sigma dx)$, we therefore obtain that $\mathcal{E}^{\tilde a}(u_n, u_n) \to 0$ as $n \to \infty$ and by (1.5) that $\|\nabla u_n\| \to 0$ as $n \to \infty$ in $L^2(U;\rho dx)$. If $(u_{n_k})_{k\in I\!N}$ is as in the claim, Fatou's lemma implies that for all $n \in I\!N$

$$\mathcal{E}_\alpha(u_n, u_n) \le \mathcal{E}^{\tilde a}(u_n, u_n) + \frac{1}{2}\liminf_{k\to\infty} \int (u_n - u_{n_k})^2\, d\mu \tag{1.31}$$

(cf. (1.26)). Hence

$$\mathcal{E}_\alpha(u_n, u_n) \le \liminf_{k\to\infty} \mathcal{E}_\alpha(u_n - u_{n_k}, u_n - u_{n_k}) \tag{1.32}$$

which can be made arbitrarily small for large enough n. Hence $(\mathcal{E}_{\alpha_0}, C_0^\infty(U))$ is closable on $L^2(U;\sigma dx)$. To prove the claim, replacing u_n by $u_n v$ for any $v \in C_0^\infty(U)$, $v \ge 0$ we may assume that supp $[u_n] \subset K$ for some compact set $K \subset U$ and all $n \in I\!N$. By the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}\frac{1}{2}\int u_n^2\, d\mu &= \int 1_K \langle \underline{d} + \underline{b}, \nabla u_n\rangle u_n\, dx + \int u_n^2 (c + \alpha\sigma)\, dx \\ &\le \left\| 1_K \|\underline{d} + \underline{b}\|\, \rho^{-1/2} u_n \right\|_2 \, \| \, \|\nabla u\|\, \rho^{1/2} \|_2 + \|1_K (c + \alpha\sigma) u_n^2\|_1.\end{aligned} \tag{1.33}$$

Using Lemma 1.5 we see that for some $\delta, \eta \in (0, \infty)$

$$\left\| 1_K \|\underline{d} + \underline{b}\|\, \rho^{-1/2} u_n \right\|_2^2 \le \delta \int \|\nabla u_n\|^2 \rho\, dx + \eta \int u_n^2 \sigma\, dx. \tag{1.34}$$

and

$$\left\| 1_K (c + \alpha\sigma) u_n^2 \right\|_1 \le \delta \int \|\nabla u_n\|^2 \rho\, dx + (\eta + \alpha) \int u_n^2 \sigma\, dx. \tag{1.35}$$

Now the claim follows.

To prove the sector condition of $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ on $L^2(U;\sigma dx)$ we recall that by [MR 92; Chapter I, 2.1(iv)] it suffices to show that there exists $K \in (0, \infty)$ such that for all $u, v \in C_0^\infty(U)$

$$|\check{\mathcal{E}}_\alpha(u, v)| \le K\, \mathcal{E}_\alpha(u, u)^{1/2} \mathcal{E}_\alpha(v, v)^{1/2}. \tag{1.36}$$

where $\check{\mathcal{E}}_\alpha(u,v) := \frac{1}{2}(\mathcal{E}_\alpha(u,v) - \mathcal{E}_\alpha(v,u))$ is the *anti-symmetric part* of $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. But for all $u, v \in C_0^\infty(U)$

$$\begin{aligned} |\check{\mathcal{E}}_\alpha(u,v)| &\leq \sum_{i,j=1}^d \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\check{a}_{ij}\,dx + \int |\langle \underline{d} - \underline{b}, \nabla u\rangle v|\,dx \\ &\leq \sup_{x,i,j} |\check{a}_{ij}(x)\rho^{-1}(x)| \; \big\| \|\nabla u\|\,\rho^{1/2}\big\|_2 \; \big\|\|\nabla v\|\,\rho^{1/2}\big\|_2 \\ &\quad + \big\|\|\underline{d} - \underline{b}\|\,\rho^{-1/2}v\big\|_2 \; \big\|\|\nabla u\|\,\rho^{1/2}\big\|_2. \end{aligned} \tag{1.37}$$

By (1.8) and Lemma 1.5 the second summand is dominated by

$$\left(\delta \int \|\nabla v\|^2 \rho\,dx + \eta \int v^2\sigma\,dx\right)^{1/2} \left(\int \|\nabla u\|^2\rho\,dx\right)^{1/2}. \tag{1.38}$$

Hence by (1.5) and (1.6) we can find a constant K' such that for all $u, v \in C_0^\infty(U)$

$$|\check{\mathcal{E}}_\alpha(u,v)| \leq K' \big(\int \|\nabla u\|_{\tilde{a}}^2\,dx + \int u^2\sigma\,dx\big)^{1/2} \; \big(\int \|\nabla v\|_{\tilde{a}}^2\,dx + \int v^2\sigma\,dx\big)^{1/2}, \tag{1.39}$$

which by (1.26) and (1.28) implies (1.36). From the proof we see that $\alpha = \alpha_0$ if $\underline{\beta} \equiv 0$. The semi-Dirichlet property (resp. the Dirichlet property if $\overline{\beta} \equiv 0$), is proved by exactly the same arguments as those on pages 48 and 49 in [MR 92]. The form $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is by definition regular (i.e., satisfies (0.1)).

The semi-Dirichlet property (resp. the Dirichlet property if $\beta = 0$), is proved similarly to [MR 92; Chapter I, Proposition 4.7]. We need that for every $\epsilon > 0$ there exists ϕ_ϵ : $\to [-\epsilon, \infty[$ such that $\phi_\epsilon(t) = t$ for all $t \in [0, \infty[$, $0 \leq \phi_\epsilon(t_1) - \phi_\epsilon(t_2) \leq t_2 - t_1$ if $t_1 \leq t_2$, $\phi_\epsilon \circ u \in D(\mathcal{E})$, $\sup_{\epsilon>0} \mathcal{E}(\phi_\epsilon \circ u, \phi_\epsilon \circ u) < \infty$ and $\limsup_{\epsilon \to 0} \mathcal{E}(u + \phi_\epsilon \circ u, u - \phi_\epsilon \circ u) \geq 0$. The boundedness $\sup_{\epsilon>0} \mathcal{E}(\phi_\epsilon \circ u, \phi_\epsilon \circ u) < \infty$ is proved by using inequality (1.28) and reducing to the case $\beta = 0$. The limsup statement follows from arguing directly as in [MR 92; Chapter I, Proposition 4.7].

The existence of the corresponding diffusions now follows from [MOR 93; Theorem 3.8] (see also [CaMe 75]) resp. [MR 92; Chapter IV, Theorem 3.5] and [MR 92; Chapter V, Theorem 1.5] (see also [O 88] and the references quoted in [MR 92; Chapter IV, Section 7 and Chapter V, Section 3] as well as [AMR 93a,b].) Note that the proof of Theorem 1.5 in [MR 92; Chapter V] can be carried over to the case of semi-Dirichlet forms. Now the proof is complete. □

2. Perturbations of a quasi-regular Dirichlet form.

We begin with the result which says that the notion of quasi-regularity is equivalent for two equivalent Dirichlet forms $\mathcal{E}'$ and $\mathcal{E}$.

Proposition 2.1. *Suppose $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form on $L^2(E;m)$ and D is an $\tilde{\mathcal{E}}_1^{1/2}$-dense linear subspace of $D(\mathcal{E})$. Let $\mathcal{E}'$ be a positive definite, bilinear form on D such that for some $c > 0$*

$$(1/c)\,\mathcal{E}_1(u) \leq \mathcal{E}'_1(u) \leq c\,\mathcal{E}_1(u), \quad \text{for all } u \in D. \tag{2.1}$$

Then the form $(\mathcal{E}', D)$ is closable in $L^2(E;m)$ and the closure $(\mathcal{E}', D(\mathcal{E}'))$ satisfies (2.1) on all of $D(\mathcal{E}') = D(\mathcal{E})$. If for some constant $K > 0$, we have

$$|\mathcal{E}'_1(u,v)| \leq K\,\mathcal{E}_1^{1/2}(u)\,\mathcal{E}_1^{1/2}(v), \quad \text{for all } u, v \in D, \tag{2.2}$$

then $(\mathcal{E}', D(\mathcal{E}'))$ is a coercive closed form on $L^2(E;m)$. Furthermore, if $(\mathcal{E}', D(\mathcal{E}'))$ is a Dirichlet form, then $(\mathcal{E}', D(\mathcal{E}'))$ is quasi-regular if and only if $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.

Proof. The proof of all but the final sentence can be found in [MR 92; Chapter I, Proposition 3.5]. To prove the final sentence, we note that for any closed set F, the spaces $D(\mathcal{E}')_F$ and $D(\mathcal{E})_F$ coincide, as both are simply the set of $u \in D(\mathcal{E})$ which vanish m-a.e. outside of the set F. From (2.1) we see that the norms $\tilde{\mathcal{E}}_1^{1/2}$ and $(\tilde{\mathcal{E}}')_1^{1/2}$ are equivalent on $D(\mathcal{E})$, so for any increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed sets, the subspace $\cup_k D(\mathcal{E})_{F_k}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense if and only if it is $(\tilde{\mathcal{E}}')_1^{1/2}$-dense. In other words, $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest if and only if it is an $\mathcal{E}'$-nest. Therefore the notions of quasi-continuity and exceptional set are also equivalent for the two forms $\mathcal{E}$ and $\mathcal{E}'$. Since $D(\mathcal{E}) = D(\mathcal{E}')$, it follows that $(\mathcal{E}', D(\mathcal{E}'))$ is quasi-regular if and only if $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. □

In the rest of this section we let $(\mathcal{E}, D(\mathcal{E}))$ be a fixed quasi-regular Dirichlet form on $L^2(E;m)$, and we consider a number of methods for getting a new quasi-regular form from the given one.

One way to get a new Dirichlet form is to perturb $\mathcal{E}$ by adding a killing term. This idea was used by, for instance, S. Albeverio and Z.M. Ma [AM 91a] to obtain a quasi-regular Dirichlet form whose domain contains no non-zero continuous function and is therefore far from being regular. This example can also be found in [MR 92; Chapter II, Example 2(e)]. In Proposition 2.3 below, we show that adding a reasonable killing term does not affect the quasi-regularity of $\mathcal{E}$.

Definition 2.2. A positive measure μ on $(E, \mathcal{B}(E))$ is said to be $\mathcal{E}$-*smooth* if $\mu(A) = 0$ for all $\mathcal{E}$-exceptional sets $A \in \mathcal{B}(E)$, and there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact sets such that $\mu(F_k) < \infty$ for all $k \in \mathbb{N}$.

The relation between smooth measures and Radon measures has recently been clarified in [AMR 93c].

Since $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, every $u \in D(\mathcal{E})$ has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ of u [MR 92; Chapter IV, Proposition 3.3(ii)]. If $\tilde{u}$ and $\tilde{u}'$ are two $\mathcal{E}$-quasi-continuous m-versions of u, then $\tilde{u} = \tilde{u}'$ $\mathcal{E}$-q.e. [MR 92; Chapter IV, Proposition 3.3(iii)] and, since μ is smooth, $\tilde{u} = \tilde{u}'$ μ-a.e. Thus, it makes sense to define $D(\mathcal{E}^\mu)$ in the following way: $D(\mathcal{E}^\mu)$ consists of all m-classes in $D(\mathcal{E})$ whose $\mathcal{E}$-quasi-continuous m-versions are μ-square integrable. Set

$$\mathcal{E}^\mu(u,v) = \mathcal{E}(u,v) + (\tilde{u},\tilde{v})_{L^2(E;\mu)} \tag{2.3}$$

for $u, v \in D(\mathcal{E}^\mu)$. The following generalizes [MR 92; Chapter IV, Theorem 4.6] (which was taken from [AM 91b] and only proved for $(\mathcal{E}, D(\mathcal{E}))$ regular).

Proposition 2.3. *If μ is an $\mathcal{E}$-smooth measure, then $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ is a quasi-regular Dirichlet form on $L^2(E;m)$.*

Proof. We begin by showing that $D(\mathcal{E}^\mu)$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$. Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest that corresponds to the measure μ, as in the definition of a smooth measure. Since $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest we know that $\cup_{k=1}^\infty D(\mathcal{E})_{F_k}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$, and furthermore, by truncation (cf. [MR 92; Chapter I, Proposition 4.17(i)]) we see that $\cup_{k=1}^\infty (L^\infty(E;m) \cap D(\mathcal{E})_{F_k})$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$. Now, for any $u \in L^\infty(E;m) \cap D(\mathcal{E})_{F_k}$ we have $\tilde{u} = 0$ $\mathcal{E}$-q.e. on $E \setminus F_k$ and $|\tilde{u}| \le \|u\|_{L^\infty(E;m)}$ $\mathcal{E}$-q.e. on E. Since $\mu(F_k) < \infty$ we find that $\tilde{u}$ is μ-square integrable and so $u \in D(\mathcal{E}^\mu)$.

In particular $D(\mathcal{E}^\mu)$ is also L^2-dense in $L^2(E;m)$ and so $\mathcal{E}^\mu$ is densely defined. By [MR 92; Chapter I, Exercise 2.1 (iv)] and [MR 92; Chapter III, Proposition 3.5] it is easy to see that $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ is a closed coercive form with the Markov property, and so is a Dirichlet form.

Now we show that any $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ is also an $\mathcal{E}^\mu$-nest. Let $u \in D(\mathcal{E}^\mu)$ and suppose that $u \ge 0$ m-a.e. so that $\tilde{u} \ge 0$ $\mathcal{E}$-q.e. Let $u_n \in \cup_k D(\mathcal{E})_{F_k}$ be a sequence which converges to u in $\tilde{\mathcal{E}}_1^{1/2}$-norm. We can replace u_n by $v_n := (u \wedge u_n) \vee 0$ without affecting $\tilde{\mathcal{E}}_1^{1/2}$-convergence (cf. the proof of 4.17 in [MR 92; Chapter I]), and by taking a subsequence we may assume that $\tilde{v}_n \to \tilde{u}$ $\mathcal{E}$-q.e. Thus $\tilde{v}_n \to \tilde{u}$ μ-a.e. and also $0 \le \tilde{v}_n \le \tilde{u}$ μ-a.e. so that $\tilde{v}_n \to \tilde{u}$ in $L^2(E;\mu)$. Since v_n already converges to u in $\tilde{\mathcal{E}}_1^{1/2}$ we have $v_n \to u$ in $(\tilde{\mathcal{E}}^\mu)_1^{1/2}$-norm. Also $v_n = 0$ wherever $u_n = 0$ and so $v_n \in D(\mathcal{E}^\mu)_{F_k}$ for some $k \in \mathbb{N}$. For an arbitrary element $u \in D(\mathcal{E}^\mu)$, we apply this argument separately to the positive and negative parts, u^+ and u^-, and conclude that $\cup_k D(\mathcal{E}^\mu)_{F_k}$ is dense in $D(\mathcal{E}^\mu)$. Thus $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}^\mu$-nest.

Since an $\mathcal{E}$-nest is also an $\mathcal{E}^\mu$-nest, the properties (QR1) and (QR2) for $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ follow immediately from (QR1) and (QR2) for $(\mathcal{E}, D(\mathcal{E}))$. By [MR 92; Chapter IV, Proposition 3.4(i)] there is a countable collection $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ of $\mathcal{E}$-quasi-continuous functions in $D(\mathcal{E}^\mu)$ that separates the points of $E \setminus N$, where N is $\mathcal{E}$-exceptional. But then $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ are also $\mathcal{E}^\mu$-quasi-continuous and N is also $\mathcal{E}^\mu$-exceptional so (QR3) holds for $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$. □

The following definition extends one given in [RW 85] (cf. also [FST 91]).

Definition 2.4. Let $D \subseteq D(\mathcal{E})$. A positive measure μ on $\mathcal{B}(E)$ is called a *D-proper speed measure* for $(\mathcal{E}, D(\mathcal{E}))$, if it does not charge any Borel $\mathcal{E}$-exceptional set, and for all

$\mathcal{E}$-quasi-continuous m-versions $\tilde{v}$ of $v \in D$,

$$\tilde{v} = 0 \quad \mu\text{-a.e. implies} \quad \tilde{v} = 0 \quad \mathcal{E}\text{-q.e.} \tag{2.4}$$

Note that because μ does not charge any $\mathcal{E}$-exceptional set, the implication in (2.4) is independent of which $\mathcal{E}$-quasi-continuous m-version of v is chosen. Let $\mathcal{M}_{PS}(D)$ denote the set of all D-proper speed measures.

The next result shows that, if D is large enough, replacing m by a D-proper speed measure does not affect the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$.

Proposition 2.5. *Let $\mu \in \mathcal{M}_{PS}(D)$ and assume, for simplicity, that $\mathcal{E}$ is symmetric. Suppose $D \subseteq D(\mathcal{E})$ is an $\mathcal{E}_1^{1/2}$-dense Stone lattice such that $\tilde{D}$ consists of μ-square integrable functions, and that $(\mathcal{E}, \tilde{D})$ is closable on $L^2(E;\mu)$ with closure $(\mathcal{E}', D(\mathcal{E}'))$. Then*

(i) $(\mathcal{E}', D(\mathcal{E}'))$ is a symmetric Dirichlet form on $L^2(E;\mu)$.

(ii) Every $\mathcal{E}$-nest is an $\mathcal{E}'$-nest.

(iii) $(\mathcal{E}', D(\mathcal{E}'))$ is quasi-regular.

Proof. First we note that condition (2.4) guarantees that $\mathcal{E}$ is well defined on $\tilde{D}$ regarded as a subspace of $L^2(E;\mu)$. We want to show that it is also densely defined. By [MR 92; Chapter IV, Proposition 3.3(i)], the space $D(\mathcal{E})$ is separable with respect to the $\mathcal{E}_1^{1/2}$-norm. Therefore we can find a sequence $(u_n)_{n\in\mathbb{N}}$ in D which is an $\mathcal{E}_1^{1/2}$-dense set. By [MR 92; Chapter IV, Proposition 3.4(i)] if we fix a sequence of $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n$, then we have

$$\{\tilde{u}_n \,|\, n \in \mathbb{N}\} \text{ separates the points in } E \setminus N, \text{ where } N \text{ is an } \mathcal{E}\text{-exceptional set in } \mathcal{B}(E). \tag{2.5}$$

Also, there exists an $h \in D(\mathcal{E})$ with an $\mathcal{E}$-quasi-continuous m-version $\tilde{h}$ which is strictly positive $\mathcal{E}$-quasi-everywhere and a sequence $(h_n)_{n\in\mathbb{N}}$ in $\tilde{D}$ which converges $\mathcal{E}$-quasi-uniformly to h. Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest so that, on each F_k, h_n is continuous, $\tilde{h}$ is strictly positive, and $h_n \to \tilde{h}$ uniformly. By taking an even smaller nest we may also assume that $\tilde{u}_n$ is continuous on F_k for every $n, k \geq 1$, and that $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ separates the points in $\cup_k F_k$.

Define for each $k \geq 1$,

$$\mathcal{A}(F_k) := \{f \in C(F_k) \,|\, f = \tilde{u}|_{F_k} \text{ for some } \mathcal{E}\text{-quasi-continuous } m\text{-version } \tilde{u} \text{ of } u \in D\}. \tag{2.6}$$

From what we have proved so far, we know that $\mathcal{A}(F_k)$ separates points and contains a strictly positive function. Since $\mathcal{A}(F_k)$ is a subspace lattice, the Stone-Weierstrass theorem tells us that it is uniformly dense in $C(F_k)$. We conclude that $\tilde{D}|_{F_k}$ is L^2-dense in $L^2(F_k;\mu|_{F_k})$ for every k, and hence $\tilde{D}$ is L^2-dense in $L^2(E;\mu)$.

Now we prove the Markov property. For $u \in \tilde{D}$ we have $u^+ \wedge 1$ in $\tilde{D}$, so

$$\mathcal{E}'(u^+ \wedge 1) = \mathcal{E}(u^+ \wedge 1) \leq \mathcal{E}(u) = \mathcal{E}'(u). \tag{2.7}$$

This proves (0.2) for $u \in \tilde{D}$ and now we apply [MR 92; Chapter I, Proposition 4.10] to get the Markov property for $\mathcal{E}'$ on all of $D(\mathcal{E}')$.

In order to prove (ii) we need a preliminary result which says that if $u \in \tilde{D}$ with $u \geq 0$ m-a.e., and if $v \in D(\mathcal{E})$ with $v \geq 0$ m-a.e., then $u \wedge \tilde{v} \in D(\mathcal{E}')$. Suppose we are given such u and v and let $v_n \in D$ so that $v_n \to v$ in $\mathcal{E}_1^{1/2}$-norm, and $\tilde{v}_n \to \tilde{v}$ $\mathcal{E}$-q.e. By replacing v_n with v_n^+ we may suppose $v_n \geq 0$. Now $u \wedge v_n \to u \wedge v$ in $\mathcal{E}_1^{1/2}$-norm, in particular $(u \wedge v_n)_{n \in \mathbb{N}}$ is $\mathcal{E}$-Cauchy. Also $u \wedge \tilde{v}_n \to u \wedge \tilde{v}$ μ-a.e. and hence, by dominated convergence, the convergence also holds in the $L^2(\mu)$ sense. Since $(\mathcal{E}', D(\mathcal{E}'))$ is a closed form, we conclude that $u \wedge \tilde{v} \in D(\mathcal{E}')$ and $u \wedge \tilde{v}_n \to u \wedge \tilde{v}$ in $(\mathcal{E}')_1^{1/2}$-norm. This result can be localized by noting that for any closed set $F \subseteq E$,

$$\begin{aligned} D(\mathcal{E})_F &= \{v \in D(\mathcal{E}) |\ v = 0 \ \ m\text{-a.e. on } F^c\} \\ &= \{v \in D(\mathcal{E}) |\ \tilde{v} = 0 \ \ \mathcal{E}\text{-q.e. on } F^c\} \\ &\subseteq \{v \in D(\mathcal{E}) |\ \tilde{v} = 0 \ \ \mu\text{-a.e. on } F^c\}. \end{aligned} \tag{2.8}$$

Therefore, if $u \in \tilde{D}$ with $u \geq 0$ m-a.e., and if $v \in D(\mathcal{E})_F$ with $v \geq 0$ m-a.e., then $u \wedge \tilde{v} \in D(\mathcal{E}')_F$.

Now let $(F_k)_{k \in \mathbb{N}}$ be an $\mathcal{E}$-nest of compact sets in E. For $u \in \tilde{D}$ let $u_n \in \cup_k D(\mathcal{E})_{F_k}$ so that $u_n \to u$ in $\mathcal{E}_1^{1/2}$-norm and, without loss of generality, $\tilde{u}_n \to u$ $\mathcal{E}$-q.e. Set

$$v_n := (\tilde{u}_n^+ \wedge u^+) - (\tilde{u}_n^- \wedge u^-). \tag{2.9}$$

Then $v_n \in \cup_k D(\mathcal{E}')_{F_k}$ and, arguing as above we have $v_n \to u$ in $(\mathcal{E}')_1^{1/2}$-norm. Since D is already $(\mathcal{E}')_1^{1/2}$-dense in $D(\mathcal{E}')$, this shows that $(F_k)_{k \in \mathbb{N}}$ is also an $\mathcal{E}'$-nest, and (ii) is proven.

By (ii) any $\mathcal{E}$-quasi-continuous function is $\mathcal{E}'$-quasi-continuous, hence (QR2) holds for $(\mathcal{E}', D(\mathcal{E}'))$. Since (QR 1,3) hold by (ii) and (2.5) respectively, $(\mathcal{E}', D(\mathcal{E}'))$ is quasi-regular. □

Remark 2.6.

(i) If $(\mathcal{E}, D(\mathcal{E}))$ is transient in the sense of [F 80] it can be proved as in [RW 85] that $(\mathcal{E}, \tilde{D})$ is always closable on $L^2(E;\mu)$. For a nice necessary and sufficient condition on μ for the closability of $(\mathcal{E}, \tilde{D})$ on $L^2(E;\mu)$ in the locally compact regular case provided μ is a Radon measure of full support we refer to [FST 91].

(ii) The condition that D is a Stone lattice may be replaced by the condition that D is closed under composition with smooth maps on $\mathbb{R}$ which vanish at the origin.

Since $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form, we may as in Definition 0.5(i), define a subspace of $(\mathcal{E}, D(\mathcal{E}))$ by setting, for any Borel set B,

$$D(\mathcal{E})_B := \{u \in D(\mathcal{E}) \,|\, \tilde{u} = 0 \ \ \mathcal{E}\text{-q.e. on } B^c\}. \tag{2.10}$$

This is more general than Definition 0.5(i) in that the set B need not be closed. It follows from [MR 92; Chapter IV, Proposition 3.3 (iii)] that these two definitions are consistent. Now $D(\mathcal{E})_B$ is a closed subspace of $(\mathcal{E}, D(\mathcal{E}))$ and it is closed under normal contractions.

Also, $D(\mathcal{E})_B$ is a subspace of $\{u \in L^2(E;m) \,|\, u = 0 \text{ } m\text{-a.e. on } B^c\}$ which can be identified with $L^2(B; m|_B)$ in the obvious way. Therefore, if

$$D(\mathcal{E})_B \text{ is } L^2\text{-dense in } L^2(B; m|_B), \tag{2.11}$$

then $(\mathcal{E}_B, D(\mathcal{E}_B))$ is a Dirichlet form on $L^2(B; m|_B)$, where we define $\mathcal{E}_B$ as the restriction of $\mathcal{E}$ to $D(\mathcal{E})_B$. We shall prove that if (2.11) holds, then $(\mathcal{E}_B, D(\mathcal{E}_B))$ is a quasi-regular Dirichlet form, but first we need a few lemmas.

Lemma 2.7. *There exists an increasing sequence $(E_k)_{k\in\mathbb{N}}$ of compact subsets of B, so that $\cup_k D(\mathcal{E})_{E_k}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})_B$.*

Proof. Fix $u \in D(\mathcal{E})_B$ and for every $\epsilon > 0$, let $u^{(\epsilon)} := u - ((-\epsilon) \vee u) \wedge \epsilon$. Let $\tilde{u}$ be an $\mathcal{E}$-quasi-continuous m-version of u. By definition, $\tilde{u} = \tilde{u}1_B$ $\mathcal{E}$-q.e. so $\tilde{u}1_B$ is also $\mathcal{E}$-quasi-continuous. So, without loss of generality, we will assume that $\tilde{u}(z) = 0$ for all $z \in B^c$. Let $(F_k)_{k\in\mathbb{N}}$ be a nest of compacts so that $\tilde{u}|_{F_k}$ is a continuous function for each k as in Definition 0.5(iv). Then

$$F_k^\epsilon := \{z \in E \,\big|\, |\tilde{u}(z)| \geq \epsilon\} \cap F_k \tag{2.12}$$

is a compact subset of B.

For any $f \in D(\mathcal{E})$ consider the function $f_{F_k^c} \in D(\mathcal{E})$ as defined in [MR 92; Chapter III, Proposition 1.5]. Then $f_{F_k^c} \geq 0$ m-a.e, $f_{F_k^c} \geq f$ m-a.e. on F_k^c, and $f_{F_k^c} \to 0$ weakly in $(D(\mathcal{E}), \mathcal{E}_1)$ as $k \to \infty$ (cf. [MR 92, page 79]). Therefore it is possible to extract a subsequence $(k_n)_{n\in\mathbb{N}}$ so that

$$w_{f,N} := \frac{1}{N}\sum_{n=1}^{N} f_{F_{k_n}^c} \tag{2.13}$$

converges strongly to zero as $N \to \infty$ (cf. [MR 92; Chapter I, Lemma 2.12]). Once again we have $w_{f,N} \geq 0$ m-a.e. and $w_{f,N} \geq f$ m-a.e. on $F_{k_N}^c$.

Now let $\{k_n\}$ be a common subsequence so that, as above,

$$w_{u^{(\epsilon)},N} \to 0 \qquad \text{and} \qquad w_{(-u^{(\epsilon)}),N} \to 0 \tag{2.14}$$

strongly as $N \to \infty$. Let $g_N := (u^{(\epsilon)} - w_{u^{(\epsilon)},N})^+$. Since $w_{u^{(\epsilon)},N} \geq u^{(\epsilon)}$ m-a.e. on $F_{k_N}^c$ we see that g_N vanishes m-a.e. on $F_{k_N}^c$. Also, $w_{u^{(\epsilon)},N} \geq 0$ m-a.e. and $u^{(\epsilon)} = 0$ m-a.e. on $\{z \in E \,\big|\, |\tilde{u}(z)| < \epsilon\}$ so g_N also vanishes m-a.e. there. This means that $g_N = 0$ m-a.e. on the set

$$\{z \in E \,\big|\, |\tilde{u}(z)| < \epsilon\} \cup F_{k_N}^c = (F_{k_N}^\epsilon)^c. \tag{2.15}$$

Thus $g_N \in D(\mathcal{E})_{F_{k_N}^\epsilon}$. Similarly $h_N := (-u^{(\epsilon)} - w_{(-u^{(\epsilon)}),N})^+ \in D(\mathcal{E})_{F_{k_N}^\epsilon}$. Thus $g_N - h_N \in D(\mathcal{E})_{F_{k_N}^\epsilon}$ and using the strong convergence in (2.14) we conclude that $g_N - h_N \to {u^{(\epsilon)}}^+ - {u^{(\epsilon)}}^- = u^{(\epsilon)}$ as $N \to \infty$.

The above argument demonstrates the existence of a sequence $(F_k)_{k\in\mathbb{N}}$ of compact subsets of B depending on u and ϵ, so that $u^{(\epsilon)}$ belongs to the closure of $\cup_k D(\mathcal{E})_{F_k}$. We need to find a single sequence of compacts that works simultaneously for all $u \in D(\mathcal{E})_B$.

Now the metric space $D(\mathcal{E})$ is separable and hence so is the subspace $D(\mathcal{E})_B$. Let $(u_i)_{i\in\mathbb{N}}$ be $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})_B$, and let $(v_i)_{i\in\mathbb{N}}$ be an enumeration of the double sequence $(u_i^{(1/j)})_{i,j\in\mathbb{N}}$ which is again dense because $u_i^{(\epsilon)} \to u_i$ as $\epsilon \to 0$. For each $i \geq 1$, let $(F_j^i)_{j\in\mathbb{N}}$ be an increasing sequence of compact subsets of B so that v_i belongs to the closure of $\cup_j D(\mathcal{E})_{F_j^i}$, and define $E_k = \cup_{i,j\leq k} F_j^i$. This new $(E_k)_{k\in\mathbb{N}}$ is an increasing sequence of compact subsets of B. For any $u \in D(\mathcal{E})_B$ and any $\delta > 0$, let v_i so $\|v_i - u\| \leq \delta/2$ and take $j \geq 1$ and $w \in D(\mathcal{E})_{F_j^i}$ so that $\|v_i - w\| \leq \delta/2$. Then $w \in D(\mathcal{E})_{E_{i\vee j}}$ and $\|u - w\| \leq \delta$. This shows that $\cup_k D(\mathcal{E})_{E_k}$ is dense in $D(\mathcal{E})_B$. □

Remark 2.8. For future reference we note that each of the E_k sets defined above is the finite union of sets of the type shown in (2.12). It follows that for each k, there exists $u \in D(\mathcal{E})_B$ with $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ so that for all $z \in E_k$ we have $\tilde{u}(z) \geq \epsilon > 0$. By truncating from above with ϵ, and from below with 0, and then multiplying by $1/\epsilon$, we may even assume that $0 \leq \tilde{u}(z) \leq 1$ everywhere on E, and $\tilde{u}(z) = 1$ identically on E_k.

Lemma 2.9. *If $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest, then $\cup_k D(\mathcal{E})_{B\cap F_k}$ is dense in $D(\mathcal{E})_B$.*

Proof. Let $f \in D(\mathcal{E})_B$ and choose sequences $(u_n)_{n\in\mathbb{N}}$,$(v_n)_{n\in\mathbb{N}} \in \cup_k D(\mathcal{E})_{F_k}$ so that $u_n \to f^+$ and $v_n \to f^-$. Then

$$w_n := \big(f^+ \wedge (u_n)^+\big) - \big(f^- \wedge (v_n)^+\big) \in \cup_k D(\mathcal{E})_{B\cap F_k} \tag{2.16}$$

and $w_n \to f$ as $n \to \infty$. □

Corollary 2.10. *If $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest and $(E_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}_B$-nest, then $(E_k \cap F_k)_{k\in\mathbb{N}}$ is also an $\mathcal{E}_B$-nest. Therefore, an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ of $u \in D(\mathcal{E})_B$ is also $\mathcal{E}_B$-quasi-continuous.*

Proof. Fix k and let $u \in D(\mathcal{E})_{E_k}$. Applying the previous lemma with $B = E_k$, we can find $k' \geq k$ and $g \in D(\mathcal{E})_{E_k\cap F_{k'}} \subseteq D(\mathcal{E})_{E_{k'}\cap F_{k'}}$ so $\|u - g\| \leq \epsilon$. This gives the first part of the result, because we already know that $\cup_k D(\mathcal{E})_{E_k}$ is dense in $D(\mathcal{E})_B$.

For $u \in D(\mathcal{E})_B$ choose an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ and an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ so that $\tilde{u}|_{F_k}$ is continuous for each $k \geq 1$. Then $(E_k \cap F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}_B$-nest and $\tilde{u}|_{E_k\cap F_k}$ is continuous for all $k \geq 1$. So $\tilde{u}$ is $\mathcal{E}_B$-quasi-continuous. □

Proposition 2.11. *If (2.11) holds, then the Dirichlet form $(\mathcal{E}_B, D(\mathcal{E}_B))$ is quasi-regular.*

Proof. Lemma 2.7 tells us that (QR1) holds, and Corollary 2.10 shows that (QR2) holds for $(\mathcal{E}_B, D(\mathcal{E}_B))$. So it only remains to show (QR3).

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $D(\mathcal{E})$ with $\mathcal{E}$-quasi-continuous m-versions $(\tilde{u}_n)_{n\in\mathbb{N}}$, and an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ so that $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ separates points in $\cup_k F_k$. Now let $(E_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}_B$-nest as constructed in Lemma 2.7, and for each k let $v_k \in D(\mathcal{E})_B$ with an $\mathcal{E}_B$-quasi-continuous m-version $\tilde{v}_k$ so that $0 \leq \tilde{v}_k(z) \leq 1$ for all $z \in B$, and $\tilde{v}_k(z) = 1$ for all $z \in E_k$ (see Remark 2.8). Then the sequence $(F_k \cap E_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}_B$-nest, and the doubly indexed sequence $(u_n v_k)_{n,k\in\mathbb{N}}$ in $D(\mathcal{E})_B$ has $\mathcal{E}_B$-quasi-continuous m-versions $(\tilde{u}_n \tilde{v}_k)_{n,k\in\mathbb{N}}$. which separate points in $\cup_k(F_k \cap E_k)$. Since $B \setminus \cup_k(F_k \cap E_k)$ is $\mathcal{E}_B$-exceptional, this gives us (QR3). $\square$

By Theorem 0.1, Proposition 2.11 has a corresponding probabilistic counterpart, i.e., it means that the restriction of an m-sectorial standard process to a Borel subset is again an m-sectorial standard process.For the definition of m-sectorial for standard processes we refer to [MR 92; Chapter IV, Section 6].

We have proved the quasi-regularity of the form $(\mathcal{E}_B, D(\mathcal{E})_B)$ provided that it is a Dirichlet form, that is, provided (2.11) holds. The following lemma gives conditions under which (2.11) will hold, in particular, it holds for every non-empty open set in E (cf. [F 80]).

Lemma 2.12. *Suppose that U is a Borel subset of E with $m(U) > 0$, and so that for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ we have $U \cap F_k$ is open in F_k for all k. Then (2.11) holds so that $(\mathcal{E}_U, D(\mathcal{E})_U)$ is a quasi-regular Dirichlet form.*

Proof. Applying Remark 2.8 to $B = E$, we see that there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ so that for each k, the space

$$\mathcal{A}(F_k) := \{f \in C(F_k) \,\big|\, f = \tilde{u}|_{F_k} \text{ for some } \mathcal{E}\text{-quasi-continuous } m\text{-version } \tilde{u} \text{ of } u \in D(\mathcal{E})\} \tag{2.17}$$

separates points and contains the function 1. Since $\mathcal{A}(F_k)$ is a subspace lattice, the Stone-Weierstrass theorem tells us that it is uniformly dense in $C(F_k)$. Using the hypothesis on U, by taking the nest even smaller we may assume that $U \cap F_k$ is open in F_k for every k.

Let K_1 and K_2 be any two disjoint closed subsets of F_k and, let $f \in C(F_k)$ so $f = 2$ on K_1 and $f = -1$ on K_2. Choose $u \in D(\mathcal{E})$ with an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ so that $\tilde{u}|_{F_k}$ is continuous and $|\tilde{u}(z) - f(z)| \le \frac{1}{2}$ for all $z \in F_k$. Then $v = (u \wedge 0) \vee 1$ has an $\mathcal{E}$-quasi-continuous m-version $(\tilde{u} \wedge 0) \vee 1$ which is continuous on F_k, is equal to 1 on K_1, and equal to 0 on K_2.

Fix $\epsilon > 0$ and let K be any compact subset of U with with $m(K) < \infty$. Since $m(K) = m(K \cap (\cup_k F_k))$ it follows that $\|1_K - 1_{K\cap F_{k_0}}\| \le \epsilon$ for some k_0. Here, and in the remainder of the proof, the norm $\|\cdot\|$ will refer to the norm in $L^2(U; m|_U)$. Since $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest, we may choose $k_1 \ge k_0$, and $u \in D(\mathcal{E})_{F_{k_1}}$ so that $\|u - 1_{K\cap F_{k_0}}\| < \epsilon$. Setting $K_1 = K \cap F_{k_0}$ and $K_2 = U^c \cap F_{k_1}$ we may use the result in the previous paragraph to find $v \in D(\mathcal{E})$ with $\mathcal{E}$-quasi-continuous m-version $\tilde{v}$ so that $0 \le \tilde{v}(z) \le 1$ for all $z \in E$, $\tilde{v} = 1$ on $K \cap F_{k_0}$, and $\tilde{v} = 0$ on $U^c \cap F_{k_1}$. Finally set $w = uv$, so $w \in D(\mathcal{E})_U$ and

$$\begin{aligned}
&\int (w(z) - 1_{K_1}(z))^2\, m(dz) \\
&\quad = \int (uv(z) - 1_{K_1}(z))^2\, m(dz) \\
&\quad = \int_{K_1} (uv(z) - 1)^2\, m(dz) + \int_{K_1^c} (uv(z))^2\, m(dz) \\
&\quad \le \int_{K_1} (u(z) - 1)^2\, m(dz) + \int_{K_1^c} (u(z))^2\, m(dz) \\
&\quad = \|u - 1_{K_1}\| \le \epsilon.
\end{aligned} \tag{2.18}$$

Then $\|w-1_K\| \le \|w-1_{K_1}\|+\|1_K-1_{K_1}\| \le \epsilon+\epsilon = 2\epsilon$, and since ϵ is arbitrary, we conclude that 1_K can be approximated from within $D(\mathcal{E})_U$ to any degree of accuracy. Now $D(\mathcal{E})_U$ is a linear space, and the linear span of such 1_K is dense in $L^2(U;m|_U)$, so therefore we see that $D(\mathcal{E})_U$ is dense in $L^2(U;m|_U)$. □

We may now use Lemma 2.12 to prove a generalization of Proposition 2.3, where we do not assume that the measure μ is smooth. In the classical case where $(\mathcal{E}, D(\mathcal{E}))$ is associated with Brownian motion, the set U in the statement of Proposition 2.13 below can be chosen in a canonical way. We refer to recent work [Stu 93] by K.T. Sturm. For a more functional analytic approach to the problem of perturbations of Dirichlet forms by not necessarily smooth measures, while addressing the problem of quasi-regularity, we refer to Theorem 4.1 in the paper by P. Stollmann and J. Voigt ([StoV 93], see also [Sto 93]).

Proposition 2.13. *Suppose that μ is a positive measure on $(E, \mathcal{B}(E))$ so that $\mu(A) = 0$ for all $\mathcal{E}$-exceptional sets $A \in \mathcal{B}(E)$. Define the form $\mathcal{E}^\mu$ as in (2.3). If $D(\mathcal{E}^\mu)$ contains at least one non-zero function, then $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ is a quasi-regular form on some Borel subset U of E.*

Proof. Recall that $D(\mathcal{E}^\mu) = \{u \in D(\mathcal{E}) \mid \int \tilde{u}(z)^2 \, \mu(dz) < \infty\}$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $D(\mathcal{E}^\mu)$ which is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E}^\mu)$, and so that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is $L^2(E;\mu)$-dense in $D(\mathcal{E}^\mu)$. Fix $\mathcal{E}$-quasi-continuous Borel m-versions $(\tilde{u}_n)_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, and define

$$F := \{z \in E \mid \tilde{u}_n(z) = 0 \text{ for all } n\}. \tag{2.19}$$

For every $u \in D(\mathcal{E}^\mu)$ we can find a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ so that $\tilde{u}_{n_k}(z) \to \tilde{u}(z)$ $\mathcal{E}$-quasi-everywhere on E. Therefore $\tilde{u}(z) = 0$ $\mathcal{E}$-q.e. on F, and since μ does not charge the Borel $\mathcal{E}$-exceptional set $(\tilde{u}(z) \neq 0) \cap F$, also $\tilde{u}(z) = 0$ μ-a.e. on F. Define $U := E \backslash F$. Then $D(\mathcal{E}^\mu)$ can be identified with a subspace of $L^2(U;m|_U)$, and $D((\mathcal{E}_U)^\mu) = D(\mathcal{E}^\mu)$ with $(\mathcal{E}_U)^\mu = \mathcal{E}^\mu$. Since we assume that $D(\mathcal{E}^\mu)$ is non-trivial, it follows that $m(U) > 0$.

Now let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest so that $\tilde{u}_n$ is continuous on F_k for each $n, k \ge 1$. Then $U \cap F_k = \cup_n \{\tilde{u}_n \neq 0\} \cap F_k$ is an open set in F_k so we may apply Lemma 2.12 and conclude that $(\mathcal{E}_U, D(\mathcal{E}_U))$ is a quasi-regular Dirichlet form on $L^2(U;m|_U)$.

Using Proposition 2.3 we will be able to get the desired conclusion provided we can show that μ (more precisely $\mu|_U$) is $\mathcal{E}_U$-smooth. Let $(E_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}_U$-nest of compact subsets of U, and without loss of generality assume that $E_k \subseteq F_k$ for all k. By construction, at each point $z \in E_k$, there exists $u \in D(\mathcal{E}^\mu)$ with an m-version $\tilde{u}$ that is continuous on E_k and so that $\tilde{u}(z) > 0$. In fact, u is one of the members of the sequence $(u_n)_{n\in\mathbb{N}}$. Now $\{z \in E_k \mid \tilde{u}_n(z) > 1/j\}_{n,j=1}^\infty$ is an open cover of E_k and so has a finite subcover. That means there exist indices $\{n_l\}_{l=1}^N$ and $\epsilon > 0$ so that $\tilde{v} := \tilde{u}_{n_1} \vee \tilde{u}_{n_2} \vee \cdots \vee \tilde{u}_{n_N}$ satisfies $\tilde{v} > \epsilon$ on E_k. Since $\int \tilde{v}(z)^2 \, \mu(dz) < \infty$, this proves that $\mu(E_k) < \infty$, which means that μ is $\mathcal{E}_U$-smooth. □

3. Quasi-regularity of square field operator Dirichlet forms.

In this section we prove a general quasi-regularity result for Dirichlet forms which are made up of a square field operator part plus a jump part and a killing part. Our proof of quasi-regularity, in particular of (QR1), will use the nest obtained using Lemma 0.6.

Let (E, ρ) be a complete, separable metric space equipped with its Borel σ-algebra $\mathcal{B}(E)$. Let m, μ, and k be positive σ-finite measures on $(E, \mathcal{B}(E))$ and J a symmetric finite positive measure on $E \times E$.

Now we start with a core D of functions. Suppose D is a linear space of bounded, continuous, real-valued functions on E, so that D separates points in E and D is closed under composition with smooth functions which vanish at the origin. We assume that each member u of D is square integrable with respect to the measures m, μ, and k. We also assume that for $u, v \in D$, if $u = v$ m-a.e., then $u = v$ (which is the case, for example, if supp$[m]$=E). Thus $D \hookrightarrow L^2(E; m)$ is a one-to-one map and so we may regard D as a subspace of $L^2(E; m)$. Finally, we assume that D does not vanish identically at any point in E, and combined with the fact that D is a point separating algebra, this implies that D is in fact dense in $L^2(E; m)$.

Next we assume that we are given a (generalized) *square field operator* Γ. This means that $\Gamma : D \times D \to L^1(E; \mu)$ is a positive bilinear mapping, where positivity means that for each $u \in D$ we have $\Gamma(u) := \Gamma(u, u) \geq 0$ μ-a.e.

We now define a bilinear form $\mathcal{E}$ on the core D by setting, for $u, v \in D$,

$$\mathcal{E}(u, v) = \int_E \Gamma(u, v)\, d\mu + \int_{E\times E} (u(z_1) - u(z_2))(v(z_1) - v(z_2))\, J(dz_1 dz_2) + \int_E uv\, dk. \quad (3.1)$$

We assume that $(\mathcal{E}, D)$ is closable in $L^2(E; m)$ and that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form (see Section 4 below for examples). Then the map $\Gamma : D \times D \to L^1(E; \mu)$ is continuous in the $\mathcal{E}_1^{1/2}$-norm and so Γ extends to a continuous bilinear map on $D(\mathcal{E})$. For notational convenience we will continue to denote this map as Γ.

We want to give conditions on Γ under which the form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Since the space D consists of continuous functions, the condition (QR2) is automatically fulfilled. Since $E \times E$ is a separable metric space, it is strongly Lindelöf and since D separates points in E, we conclude that there is a countable set $\{u_n \,|\, n \in \mathbb{N}\}$ in D that separates points in E. Thus, (QR3) is also automatically fulfilled, so in order to prove the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$ we only need to show (QR1), that is, we need to find an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

In proving the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$, it will be convenient for us to change the base measure. Consider $(\mathcal{E}, D)$ as a bilinear form, not over $L^2(E; m)$, but over $L^2(E; m+\mu+k)$. This form is again closable and its closure $(\mathcal{E}', D(\mathcal{E}'))$ is a Dirichlet form. If we could prove that $(\mathcal{E}', D(\mathcal{E}'))$ is quasi-regular, then the measure m is certainly a D-proper speed measure for $\mathcal{E}'$, and applying Proposition 2.5 we find that $(\mathcal{E}, D(\mathcal{E}))$ is also quasi-regular. Therefore, in proving quasi-regularity we could take the base measure to be $m + \mu + k$. But for notational convenience we will relabel this new base measure as m, and assume, without loss of generality, that $\mu + k \leq m$.

In the next lemma we obtain a substitute for the representation (3.1), as this representation may not hold on the complete domain $D(\mathcal{E})$.

Lemma 3.1. *For $u \in D(\mathcal{E})$, we have*

$$\mathcal{E}(u) \leq \int \Gamma(u)\, d\mu + 4J(E \times E)\, \|u\|^2_{L^\infty(E;m)} + \int u^2\, dk. \tag{3.2}$$

Proof. We begin by assuming that $\|u\|^2_{L^\infty(E;m)} < \infty$, since the inequality is trivial otherwise. From (3.1) we see that the inequality (3.2) is true for $u \in D$. Now let $u_n \in D$ so that $\mathcal{E}_1(u - u_n) \to 0$, and let ψ be a smooth function on $I\!R$ with bounded derivative so that $\sup_{x \in I\!R} |\psi(x)| \leq \|u\|_{L^\infty(E;m)} + \epsilon$, and $\psi(x) = x$ for $|x| \leq \|u\|_{L^\infty(E;m)}$. Setting $v_n := \psi(u_n)$ we see that $v_n \in D$ and from Remark 0.4 we know that $\mathcal{E}_1(u - v_n) \to 0$. In particular, $\Gamma(v_n) \to \Gamma(u)$ in $L^1(E;\mu)$ and $v_n \to u$ in $L^2(E;k)$. Now plugging v_n into (3.2) gives

$$\mathcal{E}(v_n) \leq \int \Gamma(v_n)\, d\mu + 4J(E \times E)\, (\|u\|_{L^\infty(E;m)} + \epsilon)^2 + \int v_n^2\, dk, \tag{3.3}$$

and letting $n \to \infty$ and then $\epsilon \to 0$ gives us the required result (3.2). □

The fact that m, μ, and k are not necessarily finite measures causes problems for our calculations. In order to overcome this difficulty we will use a scaling technique that was used in [ALR 93]. Since m is σ-finite we can find a function $0 < \psi \leq 1$ so that $\int \psi^2 dm < \infty$. Let $h = G_1\psi$. It follows that $0 < h \leq 1$ m-a.e. and h is 1-excessive (cf. [MR 92; Chapter III] for a discussion of excessive functions). Define $\mathcal{E}_1^h$ on $D(\mathcal{E}_1^h) := \{u \in L^2(E; h^2 m) : uh \in D(\mathcal{E})\}$ by

$$\mathcal{E}_1^h(u,v) := \mathcal{E}_1(uh, vh), \quad u, v \in D(\mathcal{E}_1^h), \tag{3.4}$$

considered as a form over $L^2(E; h^2 m)$. Since h is 1-excessive, $(\mathcal{E}_1^h, D(\mathcal{E}_1^h))$ is a Dirichlet form, and the map $u \to uh$ defines a bijective isometry between $(\mathcal{E}_1^h, D(\mathcal{E}_1^h))$ and $(\mathcal{E}_1, D(\mathcal{E}))$. Since $h > 0$ m-a.e., it follows that, for any closed set $F \subseteq E$, the subspaces $D(\mathcal{E}_1^h)_F$ and $D(\mathcal{E})_F$ correspond to each other under this isometry. Consequently, a sequence $(F_k)_{k \in I\!N}$ is an $\mathcal{E}$-nest if and only if it is an $\mathcal{E}^h$-nest. So to prove (QR1) for $(\mathcal{E}, D(\mathcal{E}))$, it suffices to prove it for $(\mathcal{E}_1^h, D(\mathcal{E}_1^h))$. For $u, v \in D(\mathcal{E}_1^h)$, let us define $\Gamma^h(u,v) := \Gamma(uh, vh)$. Notice that from Lemma 3.1, and the fact that h is bounded and belongs to both $L^2(E;m)$ and $L^2(E;k)$, we see that there exists $c > 0$ so that for every $u \in D(\mathcal{E}_1^h)$,

$$\mathcal{E}_1^h(u) \leq \int \Gamma^h(u)\, d\mu + c\|u\|^2_{L^\infty(E;m)}. \tag{3.5}$$

There is one more condition that we must impose on the operator Γ^h: we suppose that if $u, v \in D(\mathcal{E}_1^h)$, then

$$\Gamma^h\big(u \vee v\big) \leq \Gamma^h(u) \vee \Gamma^h(v) \quad \mu\text{-a.e.} \tag{3.6}$$

The following lemma gives a way in which to check condition (3.6).

Lemma 3.2. *Suppose that if $u, v \in D$, then*

$$|\Gamma(u, \phi(v))| \leq |\Gamma(u,v)| \qquad \mu\text{-a.e.}, \tag{3.7}$$

whenever ϕ is a smooth function on $I\!R$ with $\phi(0) = 0$ and $|\phi'(x)| \le 1$. Then (3.6) holds for the operator Γ on $D(\mathcal{E})$, and hence also for Γ^h on $D(\mathcal{E}_1^h)$.

Proof. Let ϕ_n be a sequence of smooth functions on $I\!R$ satisfying $\phi_n(0) = 0$ and $|\phi_n'(x)| \le 1$, and such that $\phi_n(x) \to |x|$ as $n \to \infty$. Just as in [MR 92; Chapter I, Proposition 4.17] we can show that for any $u \in D(\mathcal{E})$, $\phi_n(u) \to |u|$ in $\mathcal{E}_1^{1/2}$-norm as $n \to \infty$. For $u, v \in D$, we apply (3.7) to ϕ_n and then let $n \to \infty$ to obtain

$$|\Gamma(u, |v|)| \le |\Gamma(u, v)| \qquad \mu\text{-a.e.} \tag{3.8}$$

But since $\phi(x) = |x|$ is a normal contraction, we can use Remark 0.4 to conclude that (3.8) can be extended to all of $D(\mathcal{E})$. Now we use the inequality (3.8), the fact that Γ is a bilinear form, and the equation $x \vee y = (1/2)\{(x+y) + |x-y|\}$ to obtain (3.6) for Γ on $D(\mathcal{E})$.

$$\begin{aligned}
\Gamma(u \vee v) &= (1/4)\{\Gamma(u+v) + 2\,\Gamma(u+v, |u-v|) + \Gamma(|u-v|)\} \\
&\le (1/4)\{\Gamma(u+v) + 2\,|\Gamma(u+v, |u-v|)| + \Gamma(|u-v|)\} \\
&\le (1/4)\{\Gamma(u+v) + 2\,|\Gamma(u+v, u-v)| + \Gamma(u-v)\} \\
&= (1/4)\{\Gamma(u+v) + 2\,|\Gamma(u) - \Gamma(v)| + \Gamma(u-v)\} \\
&= (1/4)\{\Gamma(u) + 2\Gamma(u,v) + \Gamma(v) + 2|\Gamma(u) - \Gamma(v)| + \Gamma(u) - 2\Gamma(u,v) + \Gamma(v)\} \\
&= (1/2)\{(\Gamma(u) + \Gamma(v)) + |\Gamma(u) - \Gamma(v)|\} \\
&= \Gamma(u) \vee \Gamma(v).
\end{aligned} \tag{3.9}$$

If $u, v \in D(\mathcal{E}_1^h)$, then using the positivity of h and applying inequality (3.6) for Γ on $D(\mathcal{E})$ to uh and vh, we obtain (3.6) for Γ^h. □

Remark 3.3. For any $u, v \in D(\mathcal{E}_1^h)$, applying (3.6) to the pair $-u$ and $-v$ gives us

$$\Gamma^h\big(u \wedge v\big) \le \Gamma^h(u) \vee \Gamma^h(v) \qquad \mu\text{-a.e.} \tag{3.10}$$

We recall that a metric ρ_1 on E is called *uniformly equivalent* to ρ if the identity from (E, ρ) to (E, ρ_1) and its inverse are uniformly continuous.

Theorem 3.4. *Suppose that for some countable dense set $\{x_i \,|\, i \ge 1\}$ in (E, ρ), there exists a countable collection $\{f_{ij} \,|\, i \ge 1, j \ge 1\}$ of functions in $D(\mathcal{E}_1^h)$ satisfying*

$$\sup_{i,j} \Gamma^h(f_{ij}) =: \varphi \in L^1(E; \mu), \tag{3.11}$$

and

$$\rho_1(z, x_i) = \sup_j \tilde{f}_{ij}(z) \quad \mathcal{E}_1^h\text{-q.e. } z \in E \text{ for all } i \in I\!N, \tag{3.12}$$

where ρ_1 is a bounded metric on E uniformly equivalent to ρ, and $\tilde{f}_{ij}$ is an $\mathcal{E}_1^h$-quasi-continuous h^2m-version of f_{ij}. Then the Dirichlet form $(\mathcal{E}_1^h, D(\mathcal{E}_1^h))$ satisfies (QR1), *which implies that $(\mathcal{E}, D(\mathcal{E}))$ also satisfies (QR1). We conclude that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.*

Proof. Fix the index i and for each $n \geq 1$ define the function

$$u_n(z) := \sup_{j=1}^{n} \tilde{f}_{ij}(z). \tag{3.13}$$

Then $u_n \in D(\mathcal{E}_1^h)$ and u_n is $\mathcal{E}_1^h$-quasi-continuous. Furthermore, from (3.5), (3.6), and (3.10) we see that the sequence $(u_n)_{n\in I\!N}$ is bounded in $D(\mathcal{E}_1^h)$ in the $(\mathcal{E}_1^h)^{1/2}$-norm, and even more, $\Gamma^h(u_n)$ is dominated by $\varphi \in L^1(E;\mu)$. By the Banach-Saks theorem, there exists a subsequence u_{n_k} whose averages $\frac{1}{N}\sum_{k=1}^N u_{n_k}$ converge strongly in the Hilbert space $(D(\mathcal{E}_1^h), \mathcal{E}_1^h)$. From Lemma 0.6 we know that a further subsequence of these averages must converge $\mathcal{E}_1^h$-quasi-everywhere to an $\mathcal{E}_1^h$-quasi-continuous m-version of a function in $D(\mathcal{E}_1^h)$. But, on the other hand, the original sequence u_n already converges $\mathcal{E}_1^h$-q.e. to the limit $\rho_1(z, x_i)$. Thus the function $z \mapsto \rho_1(z, x_i)$ is $\mathcal{E}_1^h$-quasi-continuous, belongs to $D(\mathcal{E}_1^h)$ and satisfies $\Gamma^h(\rho_1(\cdot, x_i)) \leq \varphi \in L^1(\mu)$.

Now for each $n \geq 1$, define the function

$$w_n(z) = \inf_{i=1}^{n} \rho_1(z, x_i). \tag{3.14}$$

Arguing as above we find that the averages y_N of a subsequence of $(w_n)_{n\in I\!N}$ converge strongly in $(D(\mathcal{E}_1^h), \mathcal{E}_1^h)$, that is,

$$y_N = \frac{1}{N}\sum_{k=1}^{N} w_{n_k} \to y \quad \text{in } D(\mathcal{E}_1^h). \tag{3.15}$$

On the other hand, the sequence $(w_n)_{n\in I\!N}$ converges pointwise to zero on E and so the limit y in (3.15) is the zero function.

By Lemma 0.6 we know that a subsequence of $(y_N)_{N\in I\!N}$ converges $\mathcal{E}_1^h$-quasi-uniformly to zero on E. But since the sequence w_n is decreasing, it follows that $(w_n)_{n\in I\!N}$ itself converges $\mathcal{E}_1^h$-quasi-uniformly to zero on E. This means that there is an $\mathcal{E}_1^h$-nest $(F_k)_{k\in I\!N}$ so that, for each $k \in I\!N$, w_n converges to zero uniformly on F_k. Fix any $k \in I\!N$ and $\delta > 0$, then choose N so $w_N \leq \delta$ on F_k. Then $\inf_{i=1}^n \rho_1(z, x_i) \leq \delta$ on F_k, or

$$F_k \subseteq \bigcup_{i=1}^{N} B(x_i, \delta), \tag{3.16}$$

where $B(\delta) = \{y \in E \,|\, \rho_1(x, y) \leq \delta\}$ is the ball centered at x, with radius δ. Since this is possible for every $\delta > 0$, we conclude that F_k is totally bounded and so, since (E, ρ_1) is complete, F_k is compact. This gives (QR1) for $(\mathcal{E}_1^h, D(\mathcal{E}_1^h))$ and hence for $(\mathcal{E}, D(\mathcal{E}))$, Since (QR2) and (QR3) already hold, we conclude that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form. □

Now that we know $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular we may use the existence results [MR 92; Chapter IV, Theorem 3.5] to construct an associated strong Markov process.

Definition 3.5. A right process $\mathbf{M}$ with state space E and transition semigroup $(p_t)_{t>0}$ is called *properly associated* with $(\mathcal{E}, D(\mathcal{E}))$ if for all $f \in B_b(E) \cap L^2(E;m)$ and all $t > 0$ we have

$$p_t f \text{ is an } \mathcal{E}\text{-quasi-continuous } m\text{-version of } T_t f, \tag{3.17}$$

where $(T_t)_{t>0}$ is the semigroup on $L^2(E;m)$ generated by $(\mathcal{E}, D(\mathcal{E}))$.

Corollary 3.6. *There exists a right process* $\mathbf{M}$ *properly associated with* $(\mathcal{E}, D(\mathcal{E}))$.

Remark 3.7. The process $\mathbf{M}$ can, in fact, be taken to be an m-tight special standard process. We refer the reader to [MR 92; Chapter IV] for definitions and more details.

4. Applications.

(a) Quasi-regular gradient-type Dirichlet forms on Banach space.

Let E be a (real) separable Banach space, and μ a finite measure on $\mathcal{B}(E)$ which charges every weakly open set. Define a linear space of functions on E by

$$\mathcal{F}C_b^\infty = \{f(l_1, \dots, l_m) \,|\, m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \dots, l_m \in E'\}. \tag{4.1}$$

Here $C_b^\infty(\mathbb{R}^m)$ denotes the space of all infinitely differentiable functions on $\mathbb{R}^m$ with all partial derivatives bounded. By the Hahn-Banach theorem, $\mathcal{F}C_b^\infty$ separates the points of E. The support condition on μ means that we can regard $\mathcal{F}C_b^\infty$ as a subspace of $L^2(E;\mu)$, and a monotone class argument shows that it is dense in $L^2(E;\mu)$. Define for $u \in \mathcal{F}C_b^\infty$ and $k \in E$,

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)|_{s=0}, \;\; z \in E. \tag{4.2}$$

Observe that if $u = f(l_1, \dots, l_m)$, then

$$\frac{\partial u}{\partial k} = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(l_1, \dots, l_m) \, {}_{E'}\langle l_i, k\rangle_E, \tag{4.3}$$

which shows us that $\partial u/\partial k$ is again a member of $\mathcal{F}C_b^\infty$. Also let us assume that there is a separable real Hilbert space $(H, \langle \, , \rangle_H)$ densely and continuously embedded into E. Identifying H with its dual H' we have that

$$E' \subset H \subset E \quad \text{densely and continuously,} \tag{4.4}$$

and ${}_{E'}\langle \, , \rangle_E$ restricted to $E' \times H$ coincides with $\langle \, , \rangle_H$. Observe that by (4.3) and (4.4), for $u \in \mathcal{F}C_b^\infty$ and fixed $z \in E$, the map $k \to (\partial u/\partial k)(z)$ is a continuous linear functional on H. Define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z), \quad k \in H. \tag{4.5}$$

Define a bilinear form on $\mathcal{F}C_b^\infty$ by

$$\mathcal{E}_\mu(u, v) = \int \langle \nabla u(z), \nabla v(z)\rangle_H \, \mu(dz). \tag{4.6}$$

Assumption 4.1. We assume that the form $\mathcal{E}_\mu$ in (4.6) is closable in $L^2(E;\mu)$.

Let $\mathcal{L}_\infty(H)$ denote the set of all bounded linear operators on H with operator norm $\| \, \|_\infty$. Suppose $z \to A(z)$, $z \in E$ is a map from E to $\mathcal{L}_\infty(H)$ such that $z \to \langle A(z)h_1, h_2\rangle_H$ is $\mathcal{B}(E)$-measurable for all $h_1, h_2 \in H$. Furthermore, assume that

$$\text{there exists } \alpha \in (0, \infty) \text{ such that } \langle A(z)h, h\rangle_H \geq \alpha \|h\|_H^2 \text{ for all } h \in H, \tag{4.7}$$

and that $\|\tilde{A}\|_\infty \in L^1(E;\mu)$ and $\|\check{A}\|_\infty \in L^\infty(E;\mu)$, where $\tilde{A} := \frac{1}{2}(A+\hat{A})$, $\check{A} := \frac{1}{2}(A-\hat{A})$ and $\hat{A}(z)$ denotes the adjoint of $A(z), z \in E$. Let $c \in L^\infty(E;\mu)$ and $b, d \in L^\infty(E \to H;\mu)$ such that for all $u \in \mathcal{F}C_b^\infty$ with $u \geq 0$,

$$\int (\langle d, \nabla u\rangle_H + cu)\, d\mu \geq 0 \quad \text{and} \quad \int (\langle b, \nabla u\rangle_H + cu)\, d\mu \geq 0. \tag{4.8}$$

Define the constant $k = \|b+d\|_{L^\infty(E;\mu)}$. For $u, v \in \mathcal{F}C_b^\infty$ let,

$$\mathcal{E}_A(u,v) = \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H\, \mu(dz), \tag{4.9}$$

and

$$\mathcal{E}(u,v) = \mathcal{E}_A(u,v) + \int u\langle d, \nabla v\rangle_H\, d\mu + \int \langle b, \nabla u\rangle_H\, v\, d\mu + \int uvc\, d\mu. \tag{4.10}$$

Then Example 3e of [MR 92; Chapter II] shows us that the forms $(\mathcal{E}_A, \mathcal{F}C_b^\infty)$ and $(\mathcal{E}, \mathcal{F}C_b^\infty)$ are closable and that their closures $(\mathcal{E}_A, D(\mathcal{E}_A))$ and $(\mathcal{E}, D(\mathcal{E}))$ are Dirichlet forms.

We would like to show that these two forms are equivalent in the sense of Proposition 2.1. To begin with we note that for $u \in \mathcal{F}C_b^\infty$ we have, for μ-almost every $z \in E$,

$$\begin{aligned} |\langle (b+d)(z), \nabla u(z)\rangle_H u(z)| &\leq \|(b+d)(z)\|_H \|\nabla u(z)\|_H |u(z)| \\ &\leq 2k\, (\|\nabla u(z)\|_H^2 + |u(z)|^2) \\ &\leq 2k\, ((1/\alpha)\langle A(z)\nabla u(z), \nabla u(z)\rangle_H + |u(z)|^2) \end{aligned} \tag{4.11}$$

and

$$|c(z)u(z)^2| \leq \|c\|_{L^\infty(E;\mu)} u(z)^2. \tag{4.12}$$

Therefore, using (4.10) and integrating with respect to μ we see that for $u \in \mathcal{F}C_b^\infty$,

$$\mathcal{E}_A(u) \leq \mathcal{E}(u) \leq k_1\, (\mathcal{E}_A)_1(u), \tag{4.13}$$

where the constant k_1 can be taken to be $\max\{(2/\alpha)k+1, 2k+\|c\|_{L^\infty(E;\mu)}\}$. By Proposition 2.1, the domains $D(\mathcal{E})$ and $D(\mathcal{E}_A)$ coincide and the form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular if and only if the form $(\mathcal{E}_A, D(\mathcal{E}_A))$ is.

In order to prove quasi-regularity of $(\mathcal{E}_A, D(\mathcal{E}_A))$ (cf. [RS 92]), we shall use Theorem 3.4, and since the measure μ is already finite we do not require the re-scaling, that is, we take $h \equiv 1$. The square field operator is given on $\mathcal{F}C_b^\infty$ by $\Gamma(u,v)(z) = \langle A(z)\nabla u(z), \nabla v(z)\rangle_H$, which clearly satisfies (3.7), and by Lemma 3.2, also condition (3.6). Now since E is a separable Banach space, there exists a countable set $(l_j)_{j\in\mathbb{N}}$ in E' such that $\|l_j\|_{E'} \leq 1$ for all $j \in \mathbb{N}$, and $\|z\|_E = \sup_j l_j(z)$ for all $z \in E$. Let φ be a bounded, smooth function on $\mathbb{R}$ such that $\varphi(0) = 0$, φ is strictly increasing, and φ' is decreasing and bounded by 1. Then $\rho_1(z,x) := \varphi(\|z-x\|_E)$ is a bounded metric on E that is uniformly equivalent with the usual metric $\rho(z,x) := \|z-x\|_E$. Let $(x_i)_{i\in\mathbb{N}}$ be a countable dense subset of E, and define for every $i, j \in \mathbb{N}$,

$$f_{ij}(z) := \varphi(l_j(z-x_i)). \tag{4.14}$$

Then $f_{ij} \in \mathcal{F}C_b^\infty$ for every $i, j \in I\!N$, and

$$\nabla f_{ij}(z) = \varphi'(l_j(z - x_i))l_j, \tag{4.15}$$

and so for μ-a.e. $z \in E$,

$$\begin{aligned} \sup_{i,j} \Gamma(f_{ij})(z) &= \sup_{i,j} \langle A(z)\nabla f_{ij}(z), \nabla f_{ij}(z)\rangle_H \\ &= \sup_{i,j} (\varphi'(l_j(z - x_i)))^2 \langle A(z)l_j, l_j\rangle_H \\ &= \sup_{i,j} (\varphi'(l_j(z - x_i)))^2 \langle \tilde{A}(z)l_j, l_j\rangle_H \\ &\le \|\tilde{A}(z)\|_\infty. \end{aligned} \tag{4.16}$$

Since $\|\tilde{A}\|_\infty$ belongs to $L^1(E;\mu)$ we see that (3.11) is satisfied. On the other hand, for every fixed $i \in I\!N$, we have

$$\begin{aligned} \sup_j f_{ij}(z) &= \sup_j \varphi(l_j(z - x_i)) = \varphi(\sup_j\, l_j(z - x_i)) \\ &= \varphi(\|z - x_i\|_E) = \rho_1(z, x_i), \end{aligned} \tag{4.17}$$

for every $z \in E$ and so (3.12) is also fulfilled. Therefore Theorem 3.4 applies and we conclude that $(\mathcal{E}_A, D(\mathcal{E}_A))$, and hence $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.

(b) An intrinsic quasi-regular Dirichlet form on the free loop space.

The results of this subsection have been first proved in [ALR 93]. Our purpose here is to show that they also be obtained from the general Theorem 3.4.

Let $g := (g_{ij})$ be a uniformly elliptic Riemannian metric with bounded derivatives over $I\!R^d$ and $\Delta_g := (det\, g)^{-1/2} \sum \frac{\partial}{\partial x_i}\big[(det\, g)^{1/2} g^{ij} \frac{\partial}{\partial x_j}\big]$ the corresponding Laplacian. Let $p_t(x, y)$, $x, y \in I\!R^d, t \ge 0$, be the associated heat kernel with respect to the Riemannian volume element. Let $W(I\!R^d)$ denote the set of all continuous paths $\omega : [0, 1] \to I\!R^d$ and let $\mathcal{L}(I\!R^d) := \{\omega \in W(I\!R^d) \,|\, \omega(0) = w(1)\}$, i.e., $\mathcal{L}(I\!R^d)$ is the free loop space over $I\!R^d$. Let P_1^x be the law of the bridge defined on $\{\omega \in \mathcal{L}(I\!R^d) \,|\, \omega(0) = \omega(1) = x\}$ coming from the diffusion on $I\!R^d$ generated by Δ_g and let

$$\mu := \int P_1^x \; p_1(x, x)\, dx \tag{4.18}$$

be the *Bismut measure* on $\mathcal{L}(I\!R^d)$ which is σ-finite but not finite. We consider $\mathcal{L}(I\!R^d)$ equipped with the Borel σ-algebra coming from the uniform norm $\|\;\|_\infty$ on $\mathcal{L}(I\!R^d)$ which makes it a Banach space. The tangent space $T_\omega\mathcal{L}(I\!R^d)$ at a loop $\omega \in \mathcal{L}(I\!R^d)$ was introduced in [JL 91] as the space of *periodical* vector fields $X_t(\omega) = \tau_t(\omega)\; h(t), t \in [0, 1]$, along ω. Here τ denotes the stochastic parallel transport associated with the Levi-Civita connection of $(\mathbb{R}^d, g)$ and h belongs to the linear space H_0 consisting of all absolutely continuous maps $h : [0,1] \to T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ such that

$$(h,h)_{H_0} := \int_0^1 |h'(s)|^2 ds + \int_0^1 |h(s)|^2 ds < \infty \tag{4.19}$$

(where $|v|^2 := g_{\omega(0)}(v,v)$) and $\tau_1(\omega)h(1) = h(0)$ with $\tau_1(\omega)$ = holonomy along ω (cf. [JL 91] for details). Note that if we consider $\mathcal{L}(\mathbb{R}^d)$ as continuous maps from S^1 to $\mathbb{R}^d$ this notion is invariant by rotations of S^1 and that (4.19) induces an inner product on $T_\omega\mathcal{L}(\mathbb{R}^d)$ which turns it into a Hilbert space. Below we shall also need the Hilbert space $H_\omega\mathcal{L}(\mathbb{R}^d)(\supset T_\omega\mathcal{L}(\mathbb{R}^d))$ with inner product $(\ ,\ )_H$ which is constructed analogously but without the holonomy condition, i.e., H_0 is replaced by H which denotes the linear space of all absolutely continuous maps $h : [0,1] \to T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ satisfying (4.19). Let $\mathcal{F}C_0^\infty$ denote the linear span of the set of all functions $u : \mathcal{L}(\mathbb{R}^d) \to \mathbb{R}$ such that there exists $k \in \mathbb{N}, f \in C_0^\infty((\mathbb{R}^d)^k), t_1, \ldots, t_k \in [0,1]$ with

$$u(\omega) = f(\omega(t_1), \ldots, \omega(t_k)), \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{4.20}$$

Note that $\mathcal{F}C_0^\infty$ is dense in $L^2(\mu) :=$ (real) $L^2(\mathcal{L}(\mathbb{R}^d);\mu)$. Let $\mathcal{F}C^\infty, \mathcal{F}C_b^\infty$ be defined correspondingly with $C^\infty((\mathbb{R}^d)^k)$ resp. $C_b^\infty((\mathbb{R}^d)^k)$ replacing $C_0^\infty((\mathbb{R}^d)^k)$. We define the directional derivative of $u \in \mathcal{F}C^\infty, u$ as in (4.20), at $\omega \in \mathcal{L}(\mathbb{R}^d)$ with respect to $X(\omega) \in H_\omega\mathcal{L}(\mathbb{R}^d)$ by

$$\begin{aligned} \partial_h u(\omega) := \partial_X u(\omega) :&= \sum_{i=1}^k d_i f(\omega(t_1), \ldots, \omega(t_k)) X_{t_i}(\omega) \\ &= \sum_{i=1}^k g_{\omega(t_i)}(\nabla_i f(\omega(t_1), \ldots, \omega(t_k)), \tau_{t_i}(\omega)h(t_i)) \end{aligned} \tag{4.21}$$

where $h \in H$ with $X(\omega) = (\tau_t(\omega)h(t))_{t\in[0,1]}$ and ∇_i resp. d_i denotes the gradient (with respect to g) resp. the differential relative to the i-th coordinate of f. We extend ∂_h to all of $\mathcal{F}C^\infty$ by linearity. Note that if we consider u as a function on $W(\mathbb{R}^d)$ then

$$\partial_X u(\omega) = \frac{d}{ds} u(\omega + sX(\omega))_{|s=0}, \ \omega \in \mathcal{L}(\mathbb{R}^d). \tag{4.22}$$

Hence $\partial_X u$ is well-defined by (4.21) (i.e., independent of the special representation of u).

Let for $u \in \mathcal{F}C^\infty$ and $\omega \in \mathcal{L}(\mathbb{R}^d)$, $\widetilde{D}u(\omega)$ be the unique element in H such that $(\widetilde{D}u(\omega), h)_H = \partial_h u(\omega)$ for all $h \in H$ and let $Du(\omega)$ be its projection onto H_0. Define for $u, v \in \mathcal{F}C_0^\infty$

$$\mathcal{E}(u,v) = \int_{\mathcal{L}(\mathbb{R}^d)} (Du, Dv)_{H_0} d\mu. \tag{4.23}$$

(cf. [AR 89,90,91] for the flat case). By our assumptions on g and (4.29) below it follows that $\mathcal{E}(u,u) < \infty$ for all $u \in \mathcal{F}C_0^\infty$. By [L 92,93], [ALR 93] the densely defined quadratic form $(\mathcal{E}, \mathcal{F}C_0^\infty)$ is closable on $L^2(\mu)$. Clearly, the closure $(\mathcal{E}, D(\mathcal{E}))$ is of the type discussed in the preceding section (see (3.1)) with core $\mathcal{F}C_0^\infty$ and

$$\Gamma(u,v) = (Du, Dv)_{H_0},\ u,v \in D(\mathcal{E}) \tag{4.24}$$

where we denote the closure of D with domain $D(\mathcal{E})$ also by D. We note that D satisfies the chain rule, in particular,

$$D\phi(u) = \phi'(u)\ D(u) \text{ for all } u \in D(\mathcal{E}). \tag{4.25}$$

Hence by Lemma 3.2, Γ and consequently also Γ^h satisfy (3.6). Here and below Γ^h, $\mathcal{E}_1^h$, and $D(\mathcal{E}_1^h)$ are defined as in Section 3. It is easy to see that

$$\mathcal{F}C_b^\infty \subset D(\mathcal{E}_1^h) \text{ and } \Gamma^h(u) \le 2(h^2\Gamma(u) + u^2\Gamma(h)). \tag{4.26}$$

Let $\varphi \in C_b^\infty(\mathbb{R})$ be an odd and increasing function such that $|\varphi| \le 2, \varphi' \le 1,\ \varphi'' \le 0$ on $[0,\infty)$, and $\varphi(x) = x$ for $x \in [-1,1]$. Let $\{s_k | k \in \mathbb{N}\}$ be a dense set of [0,1] and fix $\omega_0 \in \mathcal{L}(\mathbb{R}^d)$. Let $x^l : \mathbb{R}^d \to \mathbb{R}, 1 \le l \le d$, be the standard linear coordinates and define for $j \in \mathbb{N}$

$$u_j(\omega) := \sup_{k\le j}\sup_{l\le d} |\varphi(x^l(\omega(s_k) - \omega_0(s_k)))|\ ,\ \omega \in \mathcal{L}(\mathbb{R}^d). \tag{4.27}$$

Applying first (3.6) and then (4.26) and the chain rule for D we obtain that for μ -a.e. $\omega \in \mathcal{L}(\mathbb{R}^d)$

$$\Gamma^h(u_j)(\omega) \le 4 \sup_{k\le j}\ \sup_{l\le d}\ (h^2 \|D(x^l(\omega(s_k) - \omega_0(s_k)))\|_{H_0}^2 + \|Dh(\omega)\|_{H_0}^2). \tag{4.28}$$

For $u \in \mathcal{F}C^\infty$ we have that $\|Du\|_{H_0} \le \|\widetilde{D}u\|_H$ and if $u(\omega) = f(\omega(s_1), \ldots, \omega(s_k))$ then

$$\widetilde{D}u(\omega)(s) = \sum_{i=1}^k\ G(s,s_i)\ \tau_{s_i}(\omega)^{-1} \bigtriangledown_i f(\omega(s_1), \ldots, \omega(s_k)) \tag{4.29}$$

where G is the Green function of $-\frac{d^2}{dt^2} + 1$ with Neumann boundary conditions on [0,1], i.e.,

$$G(s,u) = \frac{e}{2(e^2-1)}\ \left(e^{u+s-1} + e^{1-(u+s)} + e^{|u-s|-1} + e^{1-|u-s|}\right). \tag{4.30}$$

Hence by our assumptions on g and by (4.29) there exists $c \in]0,\infty[$ such that for all $\omega_0 \in \mathcal{L}(\mathbb{R}^d)$

$$\Gamma^h(u_j)(\omega) \le c\,(h^2 + \|Dh\|_{H_0}^2)\ \ \mu\text{-a.e.} \tag{4.31}$$

But the function on the right hand side of (4.31) is in $L^1(E;\mu)$. Let now $\{\omega_i\,|\,i \in \mathbb{N}\}$ be a dense subset of $\mathcal{L}(\mathbb{R}^d)$ and define $f_{ij} := u_{ij} \wedge 1$ where u_{ij} is equal to u_j with ω_0 replaced by $\omega_i,\ i,j \in \mathbb{N}$. Then all assumptions of Theorem 3.4 are fulfilled with the metric $\rho(\omega,\omega') := 1 \wedge \|\omega-\omega'\|_\infty,\ \omega,\omega' \in \mathcal{L}(I\!R^d)$. Hence $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. The corresponding Markov process is in fact a diffusion which is invariant by rotation of the loops. (cf. [ALR 93] for more details). Similarly, the tightness results in [DR 92] can be also derived from the general Theorem 3.4.

(c) Fleming-Viot processes.

Let $E := \mathcal{M}(S)$ be the space of probability measures on a Polish space S with Borel σ-algebra $\mathcal{B}(S)$. Let E be equipped with the topology of weak convergence. There exist uniformly continuous functions $(\phi_i)_{i\in I\!N}$ on S, such that $\|\phi_i\|_\infty \le 1$, and the topology on E is generated by the metric

$$\rho(\mu,\nu) = \sup_i \left| \int \phi_i \, d\mu - \int \phi_i \, d\nu \right|,\ \mu,\nu \in E, \tag{4.32}$$

and (E,ρ) is complete. Set $\langle \mu, \phi\rangle := \int \phi \, d\mu$ for $\phi \in C_b(S)$ and μ a finite positive measure on $\mathcal{B}(S)$ and let

$$\mathcal{F}C_b^\infty := \{ f(\langle \cdot, \psi_1\rangle, \dots, \langle \cdot, \psi_m\rangle) \,|\, m \in I\!N,\ \psi_i \in C_b(S), 1 \le i \le m,\ f \in C_b^\infty(I\!R^m)\}. \tag{4.33}$$

For $u = f(\langle \cdot, \psi_1\rangle, \dots, \langle \cdot, \psi_m\rangle) \in \mathcal{F}C_b^\infty$ and $x \in S$ define

$$\frac{\partial u}{\partial \epsilon_x}(\mu) := \frac{d}{ds} u(\mu + s\epsilon_x)|_{s=0},\ \mu \in \mathcal{M}(S), \tag{4.34}$$

and

$$\nabla u(\mu) := \left(\frac{\partial u}{\partial \epsilon_x}(\mu) \right)_{x\in S}. \tag{4.35}$$

For $\mu \in \mathcal{M}(S)$ and $f, g \in L^2(S;\mu)$ we also set

$$\langle f, g\rangle_\mu := \textstyle\int fg \, d\mu - \int f \, d\mu \int g \, d\mu. \tag{4.36}$$

Note that for $u \in \mathcal{F}C_b^\infty$, $\mu \in \mathcal{M}(S)$ the map $x \mapsto \partial u/\partial \epsilon_x(\mu)$ belongs to the space $L^2(S;\mu)$, i.e., $\nabla u(\mu) \in L^2(S;\mu)$, hence if m is a finite positive measure on the Borel sets $\mathcal{B}(E)$ of E we can define

$$\mathcal{E}(u,v) := \int \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu \, m(d\mu);\ u,v \in \mathcal{F}C_b^\infty. \tag{4.37}$$

Clearly, $\mathcal{F}C_b^\infty$ separates the points of E and therefore if $\operatorname{supp}[m] = S$, then $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is a densely defined positive definite symmetric bilinear form on $L^2(E;m)$. If it is closable, then its closure $(\mathcal{E}, D(\mathcal{E}))$ is clearly of the type studied in Section 3 with core $\mathcal{F}C_b^\infty$, and

$$\Gamma(u,v)(\mu) := \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu,\ \mu \in E. \tag{4.38}$$

It is easy to check that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form and by Lemma 3.2, Γ satisfies (3.6). Since $(\mathcal{M}(S), \rho)$ is separable we can find $\nu_i \in \mathcal{M}(S), i \in I\!N$, which are dense in $\mathcal{M}(S)$. Defining for $i, j \in I\!N$

$$f_{ij}(\mu) := \textstyle\int \phi_j \, d\mu - \int \phi_j \, d\nu_i,\ \mu \in E, \tag{4.39}$$

we see that $f_{ij} \in \mathcal{F}C_b^\infty$, that for $\mu \in E$

$$\begin{aligned} \Gamma(f_{ij})(\mu) &= \langle \phi_j, \phi_j \rangle_\mu \\ &\leq \int \phi_j^2 \, d\mu \leq 1 \in L^1(E; m), \end{aligned} \tag{4.40}$$

and that

$$\rho(\mu, \nu_i) = \sup_j f_{ij}(\mu) \text{ for all } \mu \in E, i \in \mathbb{N}. \tag{4.41}$$

Thus, Theorem 3.4 applies and $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form. The corresponding process is in fact a diffusion.

If m is the reversible invariant measure of the Fleming-Viot process on $\mathcal{M}(S)$ (cf. e.g. [EK 93]), then $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable and the corresponding process is just the Fleming-Viot process. For more details on this, a more general set-up including non-symmetric Dirichlet forms with state space $\mathcal{M}(S)$ (i.e., Fleming-Viot process with generalized selection), and a thorough study of the associated generating operators as well as the corresponding martingale problems we refer to [ORS 93].

5. Counterexamples.

(a) A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ that satisfies (QR1) and (QR2), but not (QR3).

Our first example is a regular Dirichlet form that is not quasi-regular. This form is defined over a separable, compact space E, so this example shows that the assumption of metrizability in Proposition 0.9 cannot be dropped. All that this example requires is a pathological measure space. It really has very little to do with the Dirichlet form, in fact, we will take $(\mathcal{E}, D(\mathcal{E}))$ to be the "zero" form. Let $X = [0, \Omega]$ be the first uncountable ordinal space with the order topology. Then X is a compact, Hausdorff space and $\mathcal{B}(X)$ consists of all subsets that are countable or have countable complement. We define a Borel measure μ by

$$\mu(B) = \begin{cases} 0, & \text{if } B \text{ is countable;} \\ 1, & \text{otherwise.} \end{cases} \tag{5.1}$$

A function u on X is measurable only if it is eventually constant on $[0, \Omega)$, that is, it is constant on an open set of the form (Λ, Ω) for some $\Lambda \in X$. We denote this left limit as $u(\Omega-)$. The space $L^2(X; \mu)$ identifies those measurable functions with the same left limit at Ω. Everything is fine, except for the fact that X is not separable. So now we use the fact that X is completely regular and embed it into the separable, compact space $E = [0, 1]^{[0,1]}$. Let μ^* be the image measure under this embedding and let $(z_j)_{j \in \mathbb{N}}$ be a countable dense set in E. Define a Borel measure m on $\mathcal{B}(E)$ by setting

$$m = \mu^* + \sum_{j=1}^{\infty} (\delta_{z_j}/2^j). \tag{5.2}$$

Now let $(\mathcal{E}, D(\mathcal{E}))$ be the zero form on $L^2(E; m)$, that is, $D(\mathcal{E}) = L^2(E; m)$ and $\mathcal{E}(u, v) = 0$ for all $u, v \in D(\mathcal{E})$. We now show that this Dirichlet form satisfies (QR1) and (QR2) but not (QR3). Since E is compact, (QR1) is trivially satisfied. We will prove (QR2) by showing that every measurable function u on E has an $\mathcal{E}$-quasi-continuous m-version. Let u be measurable and set

$$\tilde{u}(z) = \begin{cases} u(\Omega-), & \text{if } z = \Omega; \\ u(z), & \text{otherwise.} \end{cases} \tag{5.3}$$

Here $[0, \Omega]$ is regarded as a subset of E. Since $m(\{\Omega\}) = 0$, we see that $\tilde{u}$ is an m-version of u. Also, the sequence of closed sets

$$F_k = [0, \Omega] \cup \bigcup_{j=1}^{k} \{z_j\} \tag{5.4}$$

is an $\mathcal{E}$-nest, and, since u is constant on a set of the form $(\Lambda, \Omega) \subseteq E$, $\tilde{u}|_{F_k}$ is continuous for each k. This means that $\tilde{u}$ is $\mathcal{E}$-quasi-continuous and so (QR2) holds. Now we show that (QR3) fails. An exceptional set N must always be contained in a Borel set of measure zero, so $N \subseteq (\Lambda, \Omega)^c$ for some $\Lambda \in [0, \Omega)$. This means that any sequence of functions $(u_n)_{n \in \mathbb{N}}$ satisfying (QR3) must separate points in (Λ, Ω), for some Λ. Since each u_n is measurable, there exists $\Lambda_n \in (\Lambda, \Omega)$ so that u_n is constant on (Λ_n, Ω). Let $\Lambda^* = \sup_n \Lambda_n$. Because Ω

is the first uncountable ordinal, $\Lambda^* < \Omega$ and so the sequence $(u_n)_{n\in\mathbb{N}}$ fails to separate the points of (Λ^*, Ω). Thus no countable collection $\{u_n \,|\, n \in \mathbb{N}\}$ can satisfy the conditions of (QR3).

(b) A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ that satisfies (QR2) and (QR3), but not (QR1).

For our next example we take $E = [0,1)$ equipped with m= Lebesgue measure dz, and let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form associated with reflecting Brownian motion on $[0,1]$. That is,

$$\begin{aligned} D(\mathcal{E}) &= \{u \,|\, u \text{ is absolutely continuous on } (0,1) \text{ and } u' \in L^2((0,1); dz)\} \\ \mathcal{E}(u,v) &= (1/2) \int u'(z)\, v'(z)\, dz. \end{aligned} \tag{5.5}$$

In order to explain this example we need the following result.

Lemma 5.1. *If $u_k \to u$ in $\mathcal{E}_1^{1/2}$-norm and $m(z \,|\, u_k(z) = 0) > 0$ for all k, then there exists $z^* \in [0,1]$ so that the continuous version of u has a limit of zero at z^*.*

Proof. Since $m(z \,|\, u_k(z) = 0) > 0$, the continuous version $\tilde{u}_k$ of u_k must vanish at some point z_k in $[0,1)$. Therefore, for all $z \in [0,1)$,

$$\tilde{u}_k(z) = \int_{z_k}^{z} {u_k}'(s)\, ds. \tag{5.6}$$

Now by taking subsequences we may assume that $z_k \to z^* \in [0,1]$. Since $u_k \to u$ in $\mathcal{E}_1^{1/2}$-norm, the derivatives ${u_k}'$ converge in $L^2((0,1); dz)$ to u' so

$$\tilde{u_k}(z) = \int_{z_k}^{z} {u_k}'(s)\, ds \to \int_{z^*}^{z} u'(s)\, ds. \tag{5.7}$$

The right hand side of (5.7) is the continuous version of u and it clearly has limit zero at z^*. □

One consequence of this lemma is that an increasing sequence of sets $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest only if $m(F_k) = 1$ for some k. That is because the constant function 1 cannot be approximated from within $\cup_k D(\mathcal{E})_{F_k}$ otherwise. Since no compact subset of $[0,1)$ has full measure we see that (QR1) fails for $(\mathcal{E}, D(\mathcal{E}))$. The conditions (QR2) and (QR3) are easily seen to be satisfied.

(c) A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ that satisfies (QR1) and (QR3), but not (QR2).

In this example we take the same Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ as in Example (b), except we give $E = [0,1)$ the topology of the circle. Then E is compact so that (QR1) trivially holds. It is again quite easy to show that (QR3) holds, so we will only show that (QR2) fails. In fact, using Lemma 5.1 as before, we see that an increasing sequence $(F_k)_{k\in\mathbb{N}}$ can only be an $\mathcal{E}$-nest if $F_k = E$ for some k. Thus an $\mathcal{E}$-quasi-continuous function must be continuous everywhere on E. But $D(\mathcal{E})$ contains a lot of functions that do not have a version that is continuous on the circle, for example, $u(z) = z$. Therefore (QR2) fails.

In example (b), we saw a Dirichlet form which satisfies (QR2) and (QR3) but not (QR1) and so is not quasi-regular. In that example, the reason that quasi-regularity fails is that the space $E = [0, 1)$ is adequate to define the form but not as a state space for reflecting Brownian motion. The boundary point $\{1\}$ is missing from the space, and if we put it back, we get the usual quasi-regular form on $E = [0, 1]$ corresponding to reflecting Brownian motion. The following example shows that the problem of "missing boundary points" can even occur when E is a complete metric space. It is an example of a classical Dirichlet form, i.e., a form of gradient type defined on a complete linear space E, which, nevertheless, is not quasi-regular. Once again, it would be possible to embed E into an even larger space so that $(\mathcal{E}, D(\mathcal{E}))$ becomes quasi-regular, but we will not do it.

(d) A classical Dirichlet form that is not quasi-regular.

Let E be an infinite dimensional, separable Hilbert space and denote each point $z \in E$ as

$$z = (z_1, z_2, \ldots, z_i, \ldots), \tag{5.8}$$

where $(z_i)_{i\in\mathbb{N}}$ are the coordinates of z with respect to some fixed orthonormal basis. We equip E with its Hilbert topology and its Borel σ-algebra $\mathcal{B}(E)$. Let $(\sigma_i^2)_{i\in\mathbb{N}}$ be a sequence of strictly positive numbers such that $\sum_i \sigma_i^2 < \infty$, and let m be the Gaussian measure on $(E, \mathcal{B}(E))$ so that $(z_i)_{i\in\mathbb{N}}$ becomes a sequence of independent mean zero Gaussian random variables with $E(z_i^2) = \sigma_i^2$.

Define a core of functions, dense in $L^2(E; m)$, by

$$\mathcal{F}C_b^\infty = \{u \mid \exists k \geq 1, f \in C_b^\infty(\mathbb{R}^k) \text{ such that } u(z) = f(z_1, \ldots, z_k)\}. \tag{5.9}$$

For $i \geq 1$ and $u \in \mathcal{F}C_b^\infty$, with representation $u(z) = f(z_1, \ldots, z_k)$, we define the function $\partial u/\partial z_i$ by

$$(\partial u/\partial z_i)(z) = \begin{cases} (\partial f/\partial x_i)(z_1, \ldots, z_k), & \text{for } 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

Now let $(\gamma_i)_{i\in\mathbb{N}}$ be a sequence of strictly positive constants and define the form $\mathcal{E}$ by

$$\mathcal{E}(u, v) = \int_E \sum_{i=1}^\infty \gamma_i \, (\partial u/\partial z_i)(\partial v/\partial z_i) \, dm, \tag{5.11}$$

for $u, v \in \mathcal{F}C_b^\infty$. It can be shown that this is a well-defined, symmetric, bilinear form with the Markov property and that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable in $L^2(E; m)$. We denote its closure by $(\mathcal{E}, D(\mathcal{E}))$. The form $(\mathcal{E}, D(\mathcal{E}))$ is an example of a classical Dirichlet form of gradient type (cf. [AR 89] [AR 90] [S 90]), and because m is Gaussian, the form $(\mathcal{E}, D(\mathcal{E}))$ corresponds to an Ornstein-Uhlenbeck process. To get the required counterexample we will show that for some choice of constants $(\sigma_i^2)_{i\in\mathbb{N}}$ and $(\gamma_i)_{i\in\mathbb{N}}$, the form $(\mathcal{E}, D(\mathcal{E}))$ fails to satisfy (QR1). What this means is that this Ornstein-Uhlenbeck process cannot live on the space E, but must be modelled on a larger space where $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.

Our analysis begins with the observation that because of the simple product structure of the measure space $(E, \mathcal{B}(E), m)$, certain calculations can be reduced to one-dimensional problems. For a fixed index j, we let $z_j : E \to I\!R$ denote the map which sends z to its j^{th} coordinate. Let $P_j u = E(u \,|\, \sigma(z_j))$ be the conditional expectation with respect to $\sigma(z_j)$. The operator P_j is a projection in $L^2(E; m)$ and we claim that it is also a contraction in $(D(\mathcal{E}), \mathcal{E}_1^{1/2})$. For $u \in \mathcal{F}C_b^\infty$, with representation $u(z) = f(z_1, \ldots, z_k)$, we have the explicit formula

$$(P_j u)(z) = \int \cdots \int_{I\!R^{k-1}} f(x_1, \ldots, x_{j-1}, z_j, x_{j+1}, \ldots, x_k) \prod_{i \neq j} m_i(dx_i), \tag{5.12}$$

where m_i is the Gaussian measure on $I\!R$ with mean zero and variance σ_i^2. We note that $P_j u$ is a function of z_j only. This formula (5.12) also holds when f is not quite so smooth, for example when $f = g \vee h$ where $g, h \in C_b^\infty(I\!R^k)$. Using this explicit formula we find that for $u \in \mathcal{F}C_b^\infty$, the function $P_j u$ is again in $\mathcal{F}C_b^\infty$ and

$$\partial P_j u / \partial z_i = \delta_{ij}\, P_j(\partial u / \partial z_i). \tag{5.13}$$

Consequently,

$$\begin{aligned} \mathcal{E}(P_j u, P_j u) &= \sum_{i=1}^\infty \int \gamma_i (\partial P_j u / \partial z_i)^2 \, dm \\ &= \sum_{i=1}^\infty \int \gamma_i \delta_{ij} \big(P_j(\partial u / \partial z_j)\big)^2 \, dm \\ &= \gamma_j \int \big(P_j(\partial u / \partial z_j)\big)^2 \, dm \\ &\leq \gamma_j \int (\partial u / \partial z_j)^2 \, dm \\ &\leq \mathcal{E}(u, u). \end{aligned} \tag{5.14}$$

Therefore, also $\mathcal{E}_1(P_j u, P_j u) \leq \mathcal{E}_1(u, u)$ on $\mathcal{F}C_b^\infty$ and by continuity this inequality extends to all of $D(\mathcal{E})$. This shows that the image of $D(\mathcal{E})$ under P_j is the closure of

$$P_j \mathcal{F}C_b^\infty = \{ f(z_j) \,|\, f \in C_b^\infty(I\!R) \}. \tag{5.15}$$

Thus $P_j D(\mathcal{E})$ consists of all functions of the type $f(z_j)$ where f belongs to the closure of $C_b^\infty(R)$ with respect to the one-dimensional Dirichlet form $\gamma_j \int u'v' \, dm_j + \int uv \, dm_j$. In particular, f must be absolutely continuous.

Now let A be the open set $\{ z \in E \,\big|\, |z_j| > 1 \}$ and define

$$\mathcal{L}_A = \{ u \in D(\mathcal{E}) \,|\, u \geq 1 \ m\text{-a.e. on } A \}. \tag{5.16}$$

The element in $\mathcal{L}_A$ with the smallest $\mathcal{E}_1^{1/2}$-norm is written 1_A and is called the *reduite* of the function 1 on the set A (cf. [MR 92; Chapter III, Section 1]). We would like to show that $1_A \in P_j D(\mathcal{E})$, and to do so, it suffices to show that P_j maps 1_A back into $\mathcal{L}_A$.

First we note that since the map $u \to |u|$ is norm-reducing in $D(\mathcal{E})$, the function 1_A must be non-negative m-a.e. Now let $g \in C_b^\infty(I\!R)$ be a function satisfying $I_{(x \geq 1+\epsilon)} \leq g(x) \leq I_{(x \geq 1)}$. Then the function $v = g(z_j)$ belongs to $D(\mathcal{E})$ and $v \leq 1_A$ m-a.e. Let u_n be a sequence in $\mathcal{F}C_b^\infty$ which converges to 1_A in $\mathcal{E}_1^{1/2}$-norm. Then the sequence $u_n \vee v$ converges in $\mathcal{E}_1^{1/2}$-norm to $1_A \vee v = 1_A$. Using the fact that $g(x) \geq I_{(x \geq 1+\epsilon)}$ and the formula (5.12), we see that $P_j(u_n \vee v) \geq 1$ on $\{z \in E \,\big|\, |z_j| > 1+\epsilon\}$. As $n \to \infty$, the sequence $P_j(u_n \vee v)$ converges to $P_j 1_A$ and so this limit also must be greater than or equal to 1 on the set $\{z \in E \,\big|\, |z_j| > 1+\epsilon\}$. As this is true for every $\epsilon > 0$, we conclude that $P_j 1_A \geq 1$ on A, in other words, $P_j 1_A \in \mathcal{L}_A$.

Since $\|P_j 1_A\|_{\mathcal{E}_1^{1/2}} \leq \|1_A\|_{\mathcal{E}_1^{1/2}}$ and 1_A is the unique norm-minimizing element in $\mathcal{L}_A$, we conclude that $P_j 1_A = 1_A$. Thus 1_A has an m-version which is of the form $1_A(z) = f(z_j)$ for some absolutely continuous function f. Now this function f is equal to 1 on the set $\{x \,\big|\, |x| \geq 1\}$ and for any point $x \in (-1,1)$ we have

$$1 - f(x) = f(1) - f(x) = \int_x^1 f'(y)\,dy. \tag{5.17}$$

By Cauchy-Schwarz we get

$$(1 - f(x))^2 \leq \left(\int_{-1}^1 (f'(y))^2 \varphi_j(y)\,dy\right)\left(\sup_{|y| \leq 1} 2/\varphi_j(y)\right), \tag{5.18}$$

where φ_j is the density of the Gaussian measure m_j on $I\!R$ with mean zero and variance σ_j^2. By the norm minimizing property of 1_A we obtain

$$\gamma_j \int_{-1}^1 (f'(y))^2 \varphi_j(y)\,dy = \mathcal{E}(1_A, 1_A) \leq \mathcal{E}_1(1_A, 1_A) \leq \mathcal{E}_1(1,1) = 1, \tag{5.19}$$

and combined with the formula for $\varphi_j(y)$ this leads to the bound

$$\sup_{z \in E} |1_A(z) - 1|^2 \leq \sqrt{2\pi}\,(2\sigma_j/\gamma_j)\,\exp(1/2\sigma_j^2). \tag{5.20}$$

Proposition 5.2. *If* $(\sigma_j/\gamma_j)\exp(1/2\sigma_j^2) \to 0$ *as* $j \to \infty$, *then the Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *is not quasi-regular on* $L^2(E; m)$. *In fact,* $D(\mathcal{E})_K = \{0\}$ *for any compact set* $K \subset E$ *and so (QR1) must fail to hold.*

Note. Recalling that $\sum \sigma_j^2 < \infty$, we see that choosing $\gamma_j = \exp(1/2\sigma_j^2)$ will give us coefficients satisfying the hypothesis of the proposition above. It is typical that γ_j must go to infinity very quickly in order to force $(\mathcal{E}, D(\mathcal{E}))$ not to be quasi-regular.

Proof. For $j \geq 1$ define $A_j = \{z \in E \,\big|\, |z_j| > 1\}$ and set $O_N = \cup_{j=N}^\infty A_j$. Now if K is any compact subset of E, then $z_j \to 0$ uniformly on K as $j \to \infty$ so that $K \subseteq O_N^c$ for some N. This implies that $D(\mathcal{E})_K \subseteq D(\mathcal{E})_{O_N^c}$ and so it suffices to prove that the projection P in $(D(\mathcal{E}), \mathcal{E}_1^{1/2})$ onto the space $D(\mathcal{E})_{O_N^c}$ is the zero projection. Now since $A_j \subseteq O_N \subseteq E$ for

$j \geq N$, we have $1_{A_j} \leq 1_{O_N} \leq 1$ m-a.e. On the other hand, since $(\sigma_j/\gamma_j)\exp(1/2\sigma_j^2) \to 0$ we see from (5.20) that $1_{A_j} \to 1$ uniformly on E. Therefore $1_{O_N} = 1$.

Now for any other 1-excessive function $h \in D(\mathcal{E})$ we have

$$\int h\,dm = \mathcal{E}_1(h,1) = \mathcal{E}_1(h,1_{O_N}) = \mathcal{E}_1(h_{O_N},1) = \int h_{O_N}\,dm. \tag{5.21}$$

Noting that $h \geq h_{O_N}$ m-a.e. we conclude that $h = h_{O_N}$ in $L^2(E;m)$. Thus for every 1-excessive function h we have $Ph = h - h_{O_N} = 0$. Therefore P is zero on the linear span of all 1-excessive functions, and since this linear span is dense, we conclude that $P = 0$ on $D(\mathcal{E})$ which proves that $D(\mathcal{E})_K = \{0\}$. □

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