

Against all odds

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In 1975, when I was a high school student, the Canadian government held its first lottery, to help cover the costs of the 1976 Olympic Games in Montreal. In those days lotteries were unheard of, and this one offered one Canadian the chance of winning a million dollars at the price of ten dollars per ticket. The actual draw was televised live, as the climax of an hour-long variety show. I was very excited. I didn't win.

Nowadays, of course, lotteries are no big deal. They have become part of our daily life, with chances to win prizes big and small on practically every day of the week. Ever since blowing ten bucks in 1975, I've been fascinated with calculating the odds in games and lotteries. I'd like to show you how to calculate the chances in some different games, and finish off with a discussion of Lotto 6-49.

In solving this kind of problem, the numbers involved are very large, but the idea is quite simple: the chance of an event is the ratio of favorable outcomes to the total number of outcomes. A branch of mathematics called combinatorics helps us count these big numbers.

The Probability Formula.

$$\text{The chance of an event} = \frac{\text{The number of favorable outcomes}}{\text{The total number of outcomes}}$$

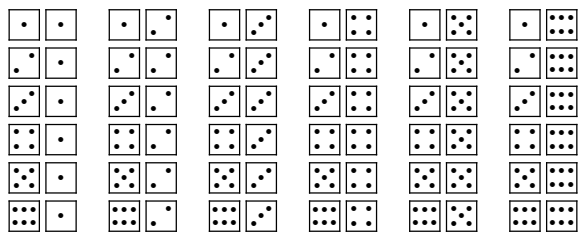
For instance, the total number of outcomes in Lotto 6-49 is 13,983,816. So, your odds of hitting the jackpot are about one in fourteen million! But how do they come up with the number 13,983,816? Did someone have to sit down with a paper and pencil, and write out all the different things that could happen?

The answer is no, to find the number of outcomes you don't have to list them out; you just use some basic mathematics. Let me show you, but before we try to tackle the Lotto 6-49 problem we'll warm up with some simpler problems.

Rolling the dice.

What's the chance of rolling a $\begin{array}{|c|} \hline \bullet\bullet\bullet \\ \hline \end{array}$, if you roll a fair die? Since all possible outcomes are $\begin{array}{|c|} \hline \bullet \\ \hline \end{array}$, $\begin{array}{|c|} \hline \bullet\bullet \\ \hline \end{array}$, $\begin{array}{|c|} \hline \bullet\bullet\bullet \\ \hline \end{array}$, $\begin{array}{|c|} \hline \bullet\bullet\bullet\bullet \\ \hline \end{array}$, $\begin{array}{|c|} \hline \bullet\bullet\bullet\bullet\bullet \\ \hline \end{array}$, $\begin{array}{|c|} \hline \bullet\bullet\bullet\bullet\bullet\bullet \\ \hline \end{array}$, and there is only one favorable outcome $\begin{array}{|c|} \hline \bullet\bullet\bullet \\ \hline \end{array}$, the chance is 1/6.

What's the chance of rolling at least one $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ if you roll two fair dice? The set of possible outcomes is illustrated here:



Counting these up we find that there are 36 possible outcomes. The outcomes that have at least one $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ are $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$. Since there are eleven of these, the chance of rolling at least one $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ with two fair dice is $11/36$.

If we want to go any farther, we need ways of counting the number of outcomes without actually listing them all out. Luckily, the multiplication rule comes to the rescue.

The multiplication rule. The number of pairs is equal to the number of choices for the first object times the number of choices for the second object. This also works for triples, quadruples, etc.

The multiplication rule says that, since there are 6 choices for each roll, the number of pairs is $6 \times 6 = 36$. Hey! This is exactly how many we counted above. How about the number of pairs with at least one $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$? Here we can't apply the multiplication rule directly. Let's look at the opposite problem, and count the number of pairs with no sixes. If we don't allow $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$, we have five choices for the first die and five for the second, so there are $5 \times 5 = 25$ pairs with no $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ s. So the number of pairs with at least one six must be $36 - 25 = 11$. The chance of getting at least one six with a pair of dice is $11/36$.

Do you see what happened? We just solved the two dice problem without listing the outcomes. The nice thing is that the same idea works no matter how many dice we use.

Suppose we roll ten dice. The total number of outcomes is $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 = 60,466,176$. The number of outcomes with no $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ s is $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 9,765,625$, so there are $60,466,176 - 9,765,625 = 50,700,551$ outcomes with at least one $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$. The chance of rolling at least one $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ with ten dice is $50,700,551/60,466,176 \approx .8385$: about an 84% chance.

Birthdays.

Do you know anyone in your class that has the same birthday as you? You might be surprised to find that matching birthdays are not that unusual. Let's figure out the chances of a shared birthday.

I am going to use the number 30 as the class size; you could find the chance in the same way for any other class size.

We ignore leap years and assume that there are 365 possible birthdays. An outcome is just a list of 30 birthdays, so the multiplication rule says that the total number of outcomes is $365^{30} = 7.392 \times 10^{76}$ points.

The number of outcomes without a shared birthday is $365 \times 364 \times 363 \times \cdots \times 336 = 2.171 \times 10^{76}$. Why? There are 365 choices for the first person's birthday. Since the second person's birthday must be different, there are only 364 choices. The third person's birthday must be different than both of the first two, leaving 363 choices, etc.

By subtracting we find that the number of outcomes with a shared birthday is $7.392 \times 10^{76} - 2.171 \times 10^{76} = 5.221 \times 10^{76}$, and so the chance of a shared birthday is $(5.221 \times 10^{76}) / (7.392 \times 10^{76}) = .7036$, about 70%.

If you wonder where the formulas come from...

The number of ways to order n distinct objects is called " n factorial" and is written $n! = n \times (n - 1) \times \cdots \times 1$. If you don't use the whole set, the number of ordered sets of r objects chosen from n distinct objects is $n \times (n - 1) \times \cdots \times (n - r + 1) = n! / (n - r)!$.

Each unordered set of size r corresponds to $r!$ distinct ordered sets, so we must divide by $r!$ to get the number of such unordered sets. This value is called " n choose r " and written $\binom{n}{r} = n! / r!(n - r)!$.

For instance, suppose that $r = 3$. Then all six ordered sets ABC , ACB , BAC , BCA , CAB , CBA correspond to the same unordered set $\{A, B, C\}$. The number of unordered sets is one-sixth the number of ordered sets.

The 52-card shuffle.

Shuffling a deck of cards puts them into a particular order. What is the probability that someone else, sometime in the past, has shuffled a deck of cards into the exact same order? The number of different ways to order a deck of cards is $52! = 8.0658 \times 10^{67}$. To understand how big this number is, let us suppose that everyone on earth, from the beginning of time, has been doing nothing but shuffling cards and checking what order the deck is in. Let's overdo it, and suppose that mankind has been on earth for a million years, and that the population of the earth was constantly equal to 10 billion. If we can do one shuffle per second, then the total number of shuffles in history is $10,000,000,000 \times 1,000,000 \times 365 \times 24 \times 60 \times 60 = 3.1536 \times 10^{23}$. This is less than one billionth of one billionth of one billionth of the total number of possible shuffles. By shuffling an ordinary deck of cards, you've created something that has never existed before in the history of the universe!

Try this at home!

Now that I've convinced you that the number of possible shuffles for a deck of cards is incredibly large, you may find the following game quite interesting. Take two decks of cards and shuffle both of them thoroughly. Give one deck to a friend and place both your decks face down. Now, at the same time, you and your friend turn over your top card. Are they the same card? No? Then try again with the second card, the third card, etc. If you go through the whole deck, do you think you and your friend will ever turn over the same card? Try it and see!

Lotto 6-49.

The most popular lottery in Canada is Lotto 6-49. Six numbers are randomly chosen from 1 to 49 and your prize depends on how many of these match the numbers on your ticket. If you match three numbers you win ten dollars, and if you match all six numbers you win the jackpot. Of course, there are other prizes for matching 4 or 5 numbers, as well. What are your chances?

The number of possible ticket combinations is $\binom{49}{6} = 13,983,816$. Your chance of winning the jackpot therefore is one out of 13,983,816, which is 7.15×10^{-8} .

As for matching three numbers, consider the numbers from 1 to 49 as divided into two groups: the six numbers on your ticket, and the forty-three numbers that aren't on your ticket. To win ten dollars you need exactly three from the first group, and three from the second group. The number of Lotto 6-49 drawings of that type is

$$\binom{6}{3} \times \binom{43}{3} = \frac{(6)(5)(4)}{(3)(2)(1)} \times \frac{(43)(42)(41)}{(3)(2)(1)} = 20 \times 12,341 = 246,820.$$

Thus, the chance of matching exactly three numbers is $246,820/13,983,816 = .017650$.

We can find all the Lotto 6-49 probabilities in the same way. The bottom of the ratio is always equal to the total number of Lotto 6-49 draws: $\binom{49}{6}$. The top of the ratio always has two terms, 43 choose something times 6 choose something. The term with 43 represents the number of ways to choose from the 43 values not on your ticket, and the other term represents the number of ways to choose from the 6 values on your ticket. If you think of the numbers as "good" or "bad" according to whether they are on your ticket or not, then $\binom{43}{6} \binom{6}{0}$ is the number of draws that result in 6 bad numbers and 0 good numbers. Similarly, $\binom{43}{5} \binom{6}{1}$ is 5 bad numbers and 1 good number, and so on. The following table gives the complete lowdown on Lotto 6-49.

Matches	Probability
0	$\frac{\binom{43}{6}\binom{6}{0}}{\binom{49}{6}} = .43596\ 49755$
1	$\frac{\binom{43}{5}\binom{6}{1}}{\binom{49}{6}} = .41301\ 94505$
2	$\frac{\binom{43}{4}\binom{6}{2}}{\binom{49}{6}} = .13237\ 80290$
3	$\frac{\binom{43}{3}\binom{6}{3}}{\binom{49}{6}} = .01765\ 04039$
4	$\frac{\binom{43}{2}\binom{6}{4}}{\binom{49}{6}} = .00096\ 86197$
5	$\frac{\binom{43}{1}\binom{6}{5}}{\binom{49}{6}} = .00001\ 84499$
6	$\frac{\binom{43}{0}\binom{6}{6}}{\binom{49}{6}} = .00000\ 00715$

Adding up the first three probabilities in the table shows that there is a better than 98% chance of losing your dollar. The odds of winning ten dollars (matching three numbers) is $.01765 \approx 1/56$, so on average you spend 56 dollars to win 10 dollars. A last bit of Lotto 6-49 trivia: If you play twice a week, every week for a thousand years, the chances are better than 99% that you never, ever win the jackpot!

An unsolved Lotto 6-49 problem.

One way to win Lotto 6-49 is to buy 13,983,816 tickets: one of each type. Of course this costs \$13,983,816 and probably isn't worth it.

Suppose you'd settle for ten dollars. What's the smallest number of Lotto 6-49 tickets you need to buy to guarantee matching at least 3 numbers? Even with today's supercomputers and advanced mathematics, the answer is: Nobody knows.