Note Title 9/16/2006

Ex 8.11: Let $Y_1, Y_2, Y_n \overset{iid}{\sim} f(y|\beta)$, where

$$f_Y(y) = f_Y(y|\beta) = \begin{cases} 3\beta^3 y^{-4} & \text{for } y \geq \beta \\ 0 & \text{elsewhere}, \quad \beta > 0 \end{cases}$$

- natural choice for an estimator of β:

$$\hat{\beta} = \min(Y_1, Y_2, \ldots Y_n) = Y_{(1)}$$

From P318 or from Theorem 6.5 on P321 with $K=1$, the pdf of $Y_{(1)}$ is given by

$$g_{Y_1}(y) = n\,[1 - F_Y(y)]^{n-1} f_Y(y).$$

From $f_Y(y|\beta)$, $F_Y(y) = \int_\beta^y f_Y(t|\beta)\,dt$

$$= \int_\beta^y 3\beta^3 t^{-4}\,dt = -3\beta^3 \left[\frac{t^{-3}}{3}\right]_\beta^y = 1 - \left(\frac{\beta}{y}\right)^3$$

$$g_{Y_1}(y) = n\left(\frac{\beta}{y}\right)^{3(n-1)} 3\beta^3 y^{-4} \quad \text{for } y \geq \beta$$

$$= 3n\,\beta^{3n} y^{-(3n+1)} \quad \text{for } y \geq \beta$$

$$E\left[Y_{(1)}\right] = E\left[\min(Y_1, Y_2, \ldots Y_n)\right]$$

$$= \int_{\beta}^{\infty} y\, f_{Y_{(1)}}(y)\, dy$$

$$= \int_{\beta}^{\infty} y\, 3n\beta^{3n}\, y^{-(3n+1)}\, dy$$

$$= 3\beta^{3n} \int_{\beta}^{\infty} y^{-3n}\, dy = -3\beta^{3n} \left.\frac{y^{-3n+1}}{3n-1}\right]_{\beta}^{\infty}$$

$$E\left[Y_{(1)}\right] = \frac{3n\beta}{3n-1} \neq \beta$$

Hence, $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$ is not unbiased for β and the bias of $\hat{\beta}$ is given by $B(\hat{\beta}) = B(Y_{(1)}) = E(Y_{(1)}) - \beta$

$$= \frac{3n\beta}{3n-1} - \beta = \frac{\beta}{3n-1}.$$

Note that, we can make $Y_{(1)}$ unbiased for β.

Consider $E\left[\frac{3n-1}{3n} Y_{(1)}\right] = \frac{3n-1}{3n}\left(\frac{3n\beta}{3n-1}\right) = \beta$.

Hence, $\frac{3n-1}{3n} Y_{(1)}$ is an unbiased estimator of β.

Denote $\hat{\beta}^* = \frac{3n-1}{3n} Y_{(1)}$.

$$V(\hat{\beta}^*) = \left(\frac{3n-1}{3n}\right)^2 V(Y_{(1)}).$$

How to evaluate $V(Y_{(1)})$?

Note that $V(Y_{(1)}) = E[Y_{(1)}^2] - [E(Y_{(1)})]^2$

$$= E[Y_{(1)}^2] - \left(\frac{3n\beta}{3n-1}\right)^2.$$

Computation of $E(Y_{(1)}^2)$:

Consider $E[Y_{(1)}^2] = \int_\beta^\infty y^2 f_{Y_{(1)}}(y)\,dy$

$$= 3n\beta^{3n}\int_\beta^\infty y^2 y^{-(3n+1)}\,dy = \frac{3n\beta^2}{3n-2}.$$

Hence, $V(Y_{(1)}) = E\left[Y_{(1)}^2\right] - \left[E(Y_{(1)})\right]^2$

$$= \frac{3n\beta^2}{3n-2} - \frac{9n^2\beta^2}{(3n-1)^2}$$

$$= 3n\beta^2\left[\frac{1}{3n-2} - \frac{3n}{(3n-1)^2}\right]$$

$$= 3n\beta^2\left[\frac{\cancel{9n^2} - \cancel{6n} + 1 - \cancel{9n^2} + \cancel{6n}}{(3n-2)(3n-1)^2}\right]$$

$$= \frac{3n\beta^2}{(3n-2)(3n-1)^2}$$

$$\therefore \quad V(\hat{\beta}^*) = \left(\frac{3n-1}{3n}\right)^2 V(Y_{(1)})$$

$$\boxed{V(\hat{\beta}^*) = \frac{\beta^2}{3n(3n-2)}}$$

Consider $E(Y) = \int_{\beta}^{y} y f_Y(y)\,dy$

$$= 3\beta^3 \int_\beta^\infty y^{-3}\,dy = -\frac{3}{2}\beta^3 \left[y^{-2} \right]_\beta^\infty$$

$$= \frac{3}{2}\beta$$

or $$E\left[\frac{2Y}{3}\right] = \beta$$

$$\Rightarrow E\left[\frac{2}{3}Y_i\right] = \beta \text{ for all } i=1,2,\ldots n.$$

$$\Rightarrow \frac{2}{3}E(\overline{Y}) = \beta$$

$\Rightarrow$ $\hat{\beta}^{**} = \frac{2}{3}\overline{Y}$ is also unbiased for β.

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$E(Y^2) = \int_\beta^\infty y^2 f_Y(y)\,dy = 3\beta^3 \int_\beta^\infty y^{-2}\,dy$$

$$= -3\beta^3 \left[y^{-1} \right]_\beta^\infty = 3\beta^2$$

$$\therefore V(Y) = 3\beta^2 - \frac{9}{4}\beta^2 = \frac{3}{4}\beta^2$$

$$\therefore \quad V(Y_i) = \frac{3}{4}\beta^2 \quad \text{for } i = 1, 2, \dots n$$

$$\Rightarrow \quad V(\bar{Y}) = \frac{3}{4n}\beta^2$$

$$\Rightarrow \quad V(\tilde{\beta}^{**}) = \frac{\cancel{4}}{3\,\cancel{9}} \cdot \frac{\cancel{3}}{\cancel{4}n}\beta^2 = \frac{\beta^2}{3n}$$

$$V(\tilde{\beta}^{*}) = \frac{\beta^2}{3n(3n-2)}$$

$$V(\tilde{\beta}^{**}) = \frac{\beta^2}{3n} > V(\tilde{\beta}^{*})$$

Relative Efficiency of $\tilde{\beta}^{*}$ with respect to $\tilde{\beta}^{**}$ is given by

$$RE[\tilde{\beta}^{*} : \tilde{\beta}^{**}] = \frac{V(\tilde{\beta}^{**})}{V(\tilde{\beta}^{*})} = 3n-2$$

$$\geq 1$$

Method of moments:

— a simple method for finding an estimator for one or two parameters

k^{th} moment of a random variable

$$\mu_k' = E(Y^k) \quad k=1,2,\ldots$$

Suppose $Y_1, Y_2, \ldots Y_n$ is a random sample, then $m_k' = \frac{1}{n}\sum_{i=1}^{n} Y_i^k$ is a sample analogue of μ_k' for $k=1,2,\ldots$

These sample based quantities, m_k', $k=1,2,\ldots$ are called <u>sample moments</u>.

By noting that μ_k''s are functions of parameters, the method of moments can be described as follows:

Equate $\mu_k' = m_k'$; $k=1,2,\ldots$ and solve for parameters.

If there are q number of unknown parameters, then q number of independent equations are needed to estimate parameters:

That is, we need

$$\mu_k' = m_k' \qquad k = 1, 2, \ldots q.$$

Examples

1) $Y \sim Bin(n, p)$

n is known and p is an unknown parameter — only one parameter

$$E(Y) = np = \mu_1'$$

Suppose $(Y_1, Y_2, \ldots, Y_n)$ is a random sample, then

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}.$$

Equate μ_1' to m_1' and solve for p.

That is, $m\hat{p} = \bar{Y}$

or $\hat{p} = \frac{\bar{Y}}{m} = \frac{1}{nm}\sum_{i=1}^{n} Y_i$ is a moment estimator of p.

Example 2. $Y \sim N(\mu, \sigma^2)$

Two parameters case.

$$E(Y) = \mu = \mu_1'$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}\,dy$$

$$= \mu^2 + \sigma^2 \quad \text{(verify)}$$

Suppose, $(Y_1, Y_2, \dots Y_n)$ is a random sample from $N(\mu, \sigma^2)$. Then, with

$m_1' = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $m_2' = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$

we have,

$$\hat{\mu}_1' = \hat{\mu} = m_1' = \bar{Y}$$

and $\hat{\mu}_2' = \hat{\mu}^2 + \hat{\sigma}^2 = m_2' = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

and $\hat{\mu} = \bar{Y}$.

NOTE: $\hat{\mu} = \bar{Y}$ is an UBE for μ

whereas $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ is

not an UBE of σ^2.

Hence, method of moment estimators need not be unbiased.