Note Title 9/20/2006

Marginal and conditional probability distributions:

a) Let Y_1 and Y_2 be jointly discrete r.vs with probability function $p(y_1, y_2)$.

Then the marginal probability functions of Y_1 and Y_2, respectively, are given by

$$p_1(y) = \sum_{y_2} p(y_1, y_2) \text{ and } p_2(y) = \sum_{y_1} p(y_1, y_2).$$

b) Let Y_1 and Y_2 be jointly continuous r.vs with joint density function $f(y_1, y_2)$. Then, the marginal density functions of Y_1 and Y_2, respectively given by

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 \text{ and } f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1$$

If Y_1 and Y_2 are jointly discrete rvs with joint probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y)$ and $p_2(y)$, respectively, then the conditional discrete probability functions of Y_1 given Y_2 is

$$p(y_1 | y_2) = P(Y_1 = y_1 | Y_2 = y_2)$$

$$= \frac{P(Y_1 = y_1, Y_2 = y_2)}{P_2(Y_2 = y_2)}$$

$$= \frac{p(y_1, y_2)}{p_2(y_2)}, \text{ provided } p_2(y_2) > 0$$

$$p(y_2 | y_1) = \frac{p(y_1, y_2)}{p_1(y_1)}, \text{ provided } p_1(y_1) > 0.$$

Note that $\sum_{y_1} p(y_1 | y_2) = 1$

and $\sum_{y_2} p(y_2 | y_1) = 1$.

Y_1 and Y_2 – jointly continuous with joint pdf $f(y_1, y_2)$; Then the conditional dist. fn of Y_1 given $Y_2 = y_2$ is given by

$$F(y_1 | y_2) = P(Y_1 \le y_1 | Y_2 = y_2)$$

$\downarrow$

function of y_1 for a fixed y_2.

$$F(y_1) = \int_{-\infty}^{\infty} F(y_1 | y_2) f_2(y_2) \, dy_2$$

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$$

$$f(y_2 | y_1) = \frac{f(y_1, y_2)}{f_1(y_1)}$$

Ex 5.1 and Ex 5.17 :

- Two jobs (contracts)
- Three firms , A, B and C

Y_1 = # of contracts awarded to A

Y_2 = - - - - - - - B

$Y_1 = 0, 1, 2$; $Y_2 = 0, 1, 2$.

Sample Space : - 9 possible cases.

$S := \{(A,A), (A,B), (B,A), (A,C), (C,A), (B,C), (C,B), (B,B), (C,C)\}$

- each one will have equal prob $= \frac{1}{9}$

y_2 \ y_1	0	1	2	marginal dist of y_2
0	1/9	2/9	1/9	4/9
1	2/9	2/9	0	4/9
2	1/9	0	0	1/9
marginal dist of y_2	4/9	4/9	1/9	

From chapter 4,

$Y_1 \sim \text{Binomial}(n=2,\ p=\frac{1}{3})$

$$P(Y_1=r) = \binom{n}{r} p^r (1-p)^{n-r}, \quad r=0,1,2$$

$r=0$, $\quad P(Y_1=0) = \binom{n}{0} p^0 (1-p)^{n-0}$

$n=2,\ p=1/3$ $\quad = \binom{2}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$$P(Y_1=1) = \binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1 = \frac{4}{9}$$

$$P(Y_1=2) = \binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^0 = \frac{1}{9}$$

no Conflict!

EX 5.7 and EX 5.23 :

Let Y_1 and Y_2 have the joint density fn given by

$$f(y_1, y_2) = \begin{cases} k(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find K. k is such that

$$\int_0^1 \left[\int_0^{y_2} f(y_1, y_2)\, dy_1\right] dy_2 = 1$$

$$\Rightarrow \quad K \int_0^1 \left[\int_0^{y_2} (1-y_2)\, dy_1\right] dy_2 = 1$$

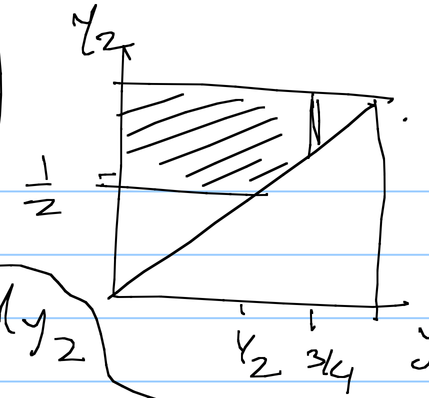
$$\Rightarrow \quad K \int_0^1 \Big[y_1(1-y_2) \Big]_0^{y_2} dy_2 = 1$$

$$\Rightarrow \quad K \int_0^1 \left[y_2(1-y_2) \right] dy_2 = 1$$

$$\Rightarrow \quad K \left[\frac{y_2^2}{2} - \frac{y_2^3}{3} \right]_0^1 = 1 \Rightarrow K\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

or $K = 6$.

b) Find $P(Y_1 \le \frac{3}{4}, Y_2 \geq 1/2)$



$$= \int_{1/2}^{1}\int_{0}^{3/4} f(y_1, y_2)\, dy_1\, dy_2$$

$$= 6\left[\int_{1/2}^{3/4}\left\{\int_{0}^{y_2}(1-y_2)\,dy_1\right\}dy_2 + \int_{3/4}^{1}\left\{\int_{0}^{3/4}(1-y_2)\,dy_1\right\}dy_2\right]$$

$$= 6\left[\int_{1/2}^{3/4}(y_2 - y_2^2)\,dy_2 + \int_{3/4}^{1}\frac{3}{4}(1-y_2)\,dy_2\right]$$

$$= \frac{31}{64}.$$

EX5.23

$$f(y_1, y_2) = \begin{cases} 6(1-y_2) & \text{for } 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

a) marginal density fns for Y_1 and Y_2.

$$f_1(y_1) = \int_{y_1}^{1} f(y_1, y_2)\, dy_2$$

$0 \le y_1 \le 1$,

$$= 6 \int_{y_1}^{1} (1-y_2)\, dy_2$$

$$= 6 \left[y_2 - \frac{y_2^2}{2} \right]_{y_1}^{1}$$

$$= 6 \left[\frac{1}{2} - \left(y_1 - \frac{y_1^2}{2} \right) \right]$$

$$= \frac{6}{2} \left(1 - 2y_1 + y_1^2 \right) = 3(1-y_1)^2.$$

$$\therefore f_1(y_1) = \begin{cases} 3(1-y_1)^2 & \text{for } 0 \le y_1 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly,

$$f_2(y_2) = \int_0^{y_2} f(y_1, y_2)\, dy_1 \quad , \; 0 \le y_2 \le 1$$

$$= 6 \int_0^{y_2} (1-y_2)\, dy_1$$

$$= 6 \; (1-y_2) \int_0^{y_2} dy_1 = 6 \; y_2 (1-y_2)$$

$$\therefore \; f_2(y_2) = \begin{cases} 6\, y_2 (1-y_2) & \text{for } 0 \le y_2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Verify (i) $\int_0^1 f_1(y_1)\, dy_1 = 1$

(ii) $\int_0^1 f_2(y_2)\, dy_2 = 1$

b) To find $P(Y_2 \le \frac{1}{2} \mid Y_1 \le \frac{3}{4})$

First find $P(Y_1 \le \frac{3}{4})$ and

$P(Y_1 \le 3/4 \, , \; Y_2 \le \frac{1}{2})$.

From $f_1(y_1) = \begin{cases} 3(1-y_1)^2 & \text{for } 0 \le y_1 \le 1 \\ 0 & \text{, elsewhere,} \end{cases}$

$$P(Y_1 \le \tfrac{3}{4}) = \int_0^{3/4} f_1(y_1)\, dy_1$$

$$= 3 \int_0^{3/4} (1-y_1)^2\, dy_1$$

$$= 3 \left[y_1 - y_1^2 + \frac{y_1^3}{3} \right]_0^{3/4} = \frac{63}{64}$$

$$P(Y_1 \le \tfrac{3}{4}, Y_2 \le \tfrac{1}{2})$$

$$= P(Y_1 \le \tfrac{1}{2}, Y_2 \le \tfrac{1}{2}) + P(\tfrac{1}{2} \le Y_1 \le \tfrac{3}{4}, Y_2 \le \tfrac{1}{2})$$

$$= P(Y_1 \le \tfrac{1}{2}, Y_2 \le \tfrac{1}{2}) + 0$$

$$= \int_0^{1/2} \left[\int_0^{y_2} f(y_1, y_2)\, dy_1 \right] dy_2$$

$$= 6 \int_0^{1/2} y_2 (1-y_2)\, dy_2 = \frac{1}{2}.$$

$\therefore \quad P(Y_2 \le \frac{1}{2} \mid Y_1 \le 3/4)$

$$= \frac{P(Y_1 \le \frac{3}{4}, Y_2 \le \frac{1}{2})}{P(Y_1 \le \frac{3}{4})}$$

$$= \frac{1/2}{\frac{63}{64}} = \frac{32}{63}.$$

c) $$f(y_1 \mid y_2 = y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} \quad ; \quad 0 < y_2 < 1$$

$$= \frac{6(1-y_2)}{6y_2(1-y_2)} = \frac{1}{y_2} \quad ; \quad 0 < y_2 < 1$$

$\therefore \quad f(y_1 \mid y_2) = \frac{1}{y_2}$ for $0 \le y_1 \le y_2$ and $0 < y_2 < 1$

d) $$f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} \quad , \quad 0 < y_1 < 1$$

$$= \frac{6(1-y_2)}{3(1-y_1)^2} = \frac{2(1-y_2)}{(1-y_1)^2} \; ; \; y_1 \le y_2 < 1, \; 0 < y_1 < 1$$

Verify $\int_0^{y_2} f(y_1 | y_2)\, dy_1 = 1$

and $\int_{y_1}^{1} f(y_2 | y_1)\, dy_2 = 1$.

e) Find $P(Y_2 \geq \frac{3}{4} \mid Y_1 = \frac{1}{2})$

$$f(y_2 | y_1) = \begin{cases} \dfrac{2(1-y_2)}{(1-y_1)^2} & \text{for } y_1 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$0 < y_1 < 1$

$$P(Y_2 \geq \tfrac{3}{4} \mid Y_1 = \tfrac{1}{2}) = \int_{3/4}^{1} f(y_2 | y_1 = \tfrac{1}{2})\, dy_2$$

$$= \int_{3/4}^{1} \frac{2(1-y_2)}{\left(\frac{1}{4}\right)}\, dy_2 = 8 \int_{3/4}^{1} (1-y_2)\, dy_2$$

$$= \frac{1}{4}.$$