Example 1 Graph the function

\[ f(x) = \begin{cases} 
\sqrt{1-x^2} & 0 \leq x < 1 \\
1 & 1 \leq x < 2 \\
2 & x = 2
\end{cases} \]

(a) What are the domain and range of \( f \)?

(b) At what points \( c \), if any, does \( \lim_{x \to c} f(x) \) exists?

(c) At what points does only the left hand limit exists?

(d) At what points does only the right hand limit exists?

Solution:
Note 2 If \( f(x) = g(x) \) when \( x \neq a \), then \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \) provided the limit exists.

Example 3 Find \( \lim_{x \to 1} g(x) \) where \( g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases} \)

Solution:

Theorem 4 If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and the limits of \( f \) and \( g \) both exists as \( x \) approaches \( a \), then

\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)
\]

Theorem 5 (The Sandwich Theorem) If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \Rightarrow \lim_{x \to a} g(x) = L \)

Example 6 Show that \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0 \).

Solution:

Example 7 Show that \( \lim_{x \to \infty} \frac{\sin(x)}{x} = 0 \)

Solution:
Note 8 \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \)

Example 9 *Calculate the following limits*

1. \( \lim_{\theta \to 0} \frac{\tan(\theta)}{\sin(3\theta)} \)
2. \( \lim_{x \to 0} \frac{x^2 - x + \sin(x)}{2x} \)
3. \( \lim_{x \to 0} \frac{\sin(5x)}{\sin(4x)} \)

**Solution:**
Section 2.6 Limits at Infinity, Asymptotes of Graphs

Definition 10 A line \( x=a \) is a vertical asymptote of the graph of a function \( y=f(x) \) if either 
\[
\lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty
\]

- For rational functions, that is of the form \( \frac{P(x)}{Q(x)} \)

\[
\lim_{x \to a} \frac{P(x)}{Q(x)} = \pm \infty \quad \text{if} \quad Q(a) = 0 \quad \text{and} \quad (x - a) \quad \text{is not a factor of} \quad P(x)
\]

Example 11 Determine the infinite limit \( \lim_{x \to 2} \frac{x^2-2x}{x^2-4x+4} \)

Solution:

Example 12 Determine the infinite limit \( \lim_{x \to 5} \frac{2x}{(x-5)^3} \)

Solution:

Definition 13 The line \( y=L \) is called a horizontal asymptote of the curve \( y=f(x) \) if either
\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L
\]

Example 14 \( \lim_{x \to \infty} \frac{1}{x} = 0 = \lim_{x \to -\infty} \frac{1}{x} \Rightarrow y=0 \) is the horizontal asymptote of \( f(x) = \frac{1}{x} \)

Theorem 15 If \( r > 0 \) is a real number and if \( x^r \) is defined for all \( x \), then \( \lim_{x \to \infty} \frac{1}{x^r} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x^r} = 0 \)
**Note 16** To find the limit of a rational function at infinity, divide both numerator and denominator by the highest power of $x$.

**Example 17** Find the following limits

1. $\lim_{x \to \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4}$
2. $\lim_{x \to \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$
3. $\lim_{x \to \infty} (\sqrt{9x^2 + x} - 3x)$
4. $\lim_{x \to \infty} \frac{x^3 + 2}{3x + 1}$
**Note 18** If a function is bounded it does not mean that function will have a horizontal asymptote.

**Example 19** \( \lim_{x \to} \sin(x) \) does not exists

**Example 20** Find horizontal and vertical asymptote of the curve \( y = \frac{2x^2 + x - 1}{x^2 + x - 2} \)
Section 2.5 Continuity

Like limits we have right and left continuity for a function

Definition 21

- A function $f$ is continuous from the right at a number $a$ if $\lim_{x \to a^+} f(x) = f(a)$.
- A function $f$ is continuous from the left at a number $a$ if $\lim_{x \to a^-} f(x) = f(a)$.
- A function $f$ is continuous at a number $a$ if $\lim_{x \to a} f(x) = f(a)$.

To check if $f$ is continuous at $x = a$

1. Check if $f$ is defined at $x = a$.
2. Check if $\lim_{x \to a} f(x)$ exists.
3. Check if $\lim_{x \to a} f(x) = f(a)$.

Note 22

If $f$ is defined on $[a, b]$ then we say $f$ is continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$, and we say $f$ is continuous at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$.

Example 23 Let

$$f(x) = \begin{cases} 
  x^2 - 1 & -1 \leq x < 0 \\
  2x & 0 < x < 1 \\
  -2x + 4 & 1 \leq x 
\end{cases}$$

Is $f$ continuous at $x = -1, x = 0, x = 1$?

Solution:
Example 24  Is $f(x) = |x|$ continuous at $x=0$?

Solution:

Theorem 25  The following type of functions are continuous at every number in their domains: Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions.

Example 26  Find the numbers at which $f$ is discontinuous.

$$f(x) = \begin{cases} 
1 + x^2 & \text{if } x \leq 0 \\
2 - x & \text{if } 0 < x \leq 2 \\
(x - 2)^2 & \text{if } x > 2
\end{cases}$$

Solution:

Definition 27  A function is continuous on an interval if it is continuous at every number in the interval.

Example 28  For what values of the constant $c$ is the function $f$ continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} 
 cx^2 + 2x & \text{if } x < 2 \\
x^3 - cx & \text{if } x \geq 2
\end{cases}$$
Theorem 29 If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

1. $f+g$
2. $f-g$
3. $cf$
4. $fg$
5. $\frac{f}{g}$ if $g(a) \neq 0$
6. $f^n$, $n$ is a positive integer
7. $\sqrt[n]{f}$, $n$ is positive

Theorem 30 If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$$

Example 31 $\lim_{x \to 0} \tan \left(\frac{\pi}{4} \cos (\sin (x^{1/3}))\right)$

Solution:

Theorem 32 If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$ then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at $x=a$. 
Example 33  At what points the function \( f(x) = \sqrt{2x + 3} \) is continuous?

Solution:

Theorem 34  (Intermediate Value Theorem)
Suppose that \( f \) is continuous on the closed interval \([a,b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \((a,b)\) such that \( f(c) = N \).

Example 35  Show that there is a root of the equation \( 4x^3 - 6x^2 + 3x - 2 = 0 \).

Solution: